

# Stabilizations of reducible Heegaard splittings

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## Abstract

Cameron Gordon([Problem 3.91 in [Ki]) conjectured that a connected sum of two Heegaard splittings is stabilized if and only if one of the two factors is stabilized. In this paper, we shall prove this conjecture.

**Keywords:** Abstract tree, band sum, connected sum, stabilization.

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## 1 Introduction

Let  $M$  be a compact 3-manifold such that  $\partial M$  has no 2-sphere components. A Heegaard splitting of  $M$  is a pair  $(\mathcal{V}, \mathcal{W})$ , where  $\mathcal{V}$  and  $\mathcal{W}$  are compression bodies such that  $\mathcal{V} \cup \mathcal{W} = M$ , and  $\mathcal{V} \cap \mathcal{W} = \partial_+ \mathcal{V} = \partial_+ \mathcal{W} = F$ .  $F$  is called a Heegaard surface of  $M$ . The splitting is often denoted as  $\mathcal{V} \cup_F \mathcal{W}$  or  $\mathcal{V} \cup \mathcal{W}$ . It is known that any compact 3-manifold has a Heegaard splitting.

Let  $\mathcal{V} \cup \mathcal{W}$  be a Heegaard splitting of  $M$ .  $\mathcal{V} \cup \mathcal{W}$  is said to be reducible if there exist essential disks  $B_{\mathcal{V}} \subset \mathcal{V}$  and  $B_{\mathcal{W}} \subset \mathcal{W}$  with  $\partial B_{\mathcal{V}} = \partial B_{\mathcal{W}}$ . Otherwise, it is said to be irreducible. W. Haken[H] showed that any Heegaard splitting of a reducible, compact 3-manifold is reducible, Kneser[Kn] and Milnor[M] showed that any orientable, compact 3-manifold is a connected sum of  $n$  irreducible 3-manifolds  $M_1, \dots, M_n$  where  $M_1, \dots, M_n$

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are unique up to isotopy. Thus any Heegaard splitting of  $M$  is the connected sums of the Heegaard splittings of  $n$  factors  $M_1, \dots, M_n$  defined as following:

Now let  $M = \mathcal{V} \cup \mathcal{W}$  be a reducible Heegaard splitting. Then  $P = B_{\mathcal{V}} \cup B_{\mathcal{W}}$  is a 2-sphere. Suppose that  $P$  cuts  $M$  into  $M_+^*$  and  $M_-^*$ . Then  $B_{\mathcal{V}}$  separates  $\mathcal{V}$  into  $\mathcal{V}_+$  and  $\mathcal{V}_-$ ,  $B_{\mathcal{W}}$  separates  $\mathcal{W}$  into  $\mathcal{W}_+^*$  and  $\mathcal{W}_-^*$ . We may assume that  $\mathcal{V}_+, \mathcal{W}_+^* \subset M_+$  and  $\mathcal{V}_-, \mathcal{W}_-^* \subset M_-$ . Let  $M_+ = M_+^* \cup_P H_+^3$  and  $M_- = M_-^* \cup_P H_-^3$  where  $H_+^3$  and  $H_-^3$  are two 3-balls. Then  $M$  is the connected sum of  $M_+$  and  $M_-$ , denoted by  $M = M_+ \sharp M_-$ . Let  $\mathcal{W}_+ = \mathcal{W}_+^* \cup H_+^3$ , and  $\mathcal{W}_- = \mathcal{W}_-^* \cup H_-^3$ . Then  $\mathcal{W}_+$  and  $\mathcal{W}_-$  are two compression bodies such that  $\partial_+ \mathcal{V}_+ = \partial_+ \mathcal{W}_+$  and  $\partial_+ \mathcal{V}_- = \partial_+ \mathcal{W}_-$ . Hence  $M_+ = \mathcal{V}_+ \cup \mathcal{W}_+$  is a Heegaard splitting of  $M_+$  and  $M_- = \mathcal{V}_- \cup \mathcal{W}_-$  is a Heegaard splitting of  $M_-$ . In this case,  $\mathcal{V} \cup \mathcal{W}$  is called the connected sum of  $\mathcal{V}_+ \cup \mathcal{W}_+$  and  $\mathcal{V}_- \cup \mathcal{W}_-$ .

A Heegaard splitting  $M = \mathcal{V} \cup \mathcal{W}$  is said to be stabilized if there are two properly embedded disks  $V \subset \mathcal{V}$  and  $W \subset \mathcal{W}$  such that  $V$  intersects  $W$  in one point; otherwise, it is said to be unstabilized. Some important results on stabilizations of Heegaard splittings have been given in [RS], [S], [ST] and [W].

An interesting problem on stabilizations of Heegaard splittings offered by C. Gordon is the following:

***Gordon's conjecture.*** The connected sum of two Heegaard splittings is stabilized if and only if one of the two factors is stabilized. (See Problem 3.91 in [Ki].)

In this paper, we shall give a proof to Gordon's conjecture. The main result is the following theorem:

***Theorem 1.*** The connected sum of two Heegaard splittings is stabilized if and only if one of the two factors is stabilized.

#### **Comments on Theorem 1.**

(1) By Haken's lemma, the connected sum of the minimal Heegaard splittings of  $M_+$  and  $M_-$  is unstabilized; but there are many manifolds which have unstabilized Heegaard splittings of distinct genera. There are examples, given by A. Casson and C. Gordon[CG1], independently by T. Kobayashi[Ko], which have irreducible Heegaard splittings of arbitrarily

high genera. Now let  $M_1$  be such a manifold, and  $M_2$  be any compact 3-manifold. Then, by Theorem 1,  $M_1 \sharp M_2$  has unstabilized Heegaard splittings of arbitrarily high genus.

(2) David Bachman[B] announced that if  $M_+$  and  $M_-$  are closed and irreducible then Gordon's conjecture is true. In this case,  $\mathcal{V}_+ \cup \mathcal{W}_+$  and  $\mathcal{V}_- \cup \mathcal{W}_-$  are irreducible.

(3) Two applications of Theorem 1 have been given in [QM]:

(a) A Heegaard splitting  $M = \mathcal{W} \cup_S \mathcal{V}$  is said to be boundary reducible if there is an essential disk  $D$  of  $M$  which intersects  $S$  in an essential simple curve in  $S$ . A. Casson and C. Gordon proved that any Heegaard splitting of a boundary reducible 3-manifold is boundary reducible in [CG2]. In [QM], we shall prove that any Heegaard splitting  $\mathcal{W} \cup \mathcal{V}$  of a boundary reducible, irreducible 3-manifold is obtained by doing boundary connected sums and self-boundary connected sums from Heegaard splittings of  $n$  3-manifolds  $M_1, \dots, M_n$ , where  $M_i$  is either boundary irreducible or a solid torus. Furthermore,  $\mathcal{W} \cup \mathcal{V}$  is unstabilized if and only if one of the factors is unstabilized. This result can be taken as the disk version of Gordon's conjecture.

(b) Suppose that  $M_1$  and  $M_2$  are two compact 3-manifolds with boundary. Let  $A_i$  be an incompressible annulus in  $\partial M_i$ , and  $M = M_1 \cup_{A_1=A_2} M_2$ . Let  $M_i = \mathcal{W}_i \cup \mathcal{V}_i$  be an unstabilized Heegaard splitting of  $M_i$ . Then  $M$  has a natural Heegaard splitting  $\mathcal{W} \cup \mathcal{V}$  induced by  $\mathcal{W}_1 \cup \mathcal{V}_1$  and  $\mathcal{W}_2 \cup \mathcal{V}_2$  such that  $g(\mathcal{W}) = g(\mathcal{W}_1) + g(\mathcal{W}_2)$ . Without loss of generality, we may assume that  $A_1 \subset \partial_- \mathcal{V}_i$ . We denote by  $M_i(A_i)$  the manifold obtained by attaching a 2-handle to  $M_i$  along  $A_i$ ,  $\mathcal{V}_i(A_i)$  the manifold obtained by attaching a 2-handle to  $\mathcal{V}_i$  along  $A_i$ . Then  $M_i(A_i) = \mathcal{W}_i \cup \mathcal{V}_i(A_i)$  is a Heegaard splitting of  $M_i(A_i)$ . In [QM], we shall prove that if  $M_1(A_1) = \mathcal{W}_1 \cup \mathcal{V}_1(A_1)$  and  $M_2(A_2) = \mathcal{W}_2 \cup \mathcal{V}_2(A_2)$  are unstabilized, then  $\mathcal{W} \cup \mathcal{V}$  is unstabilized.

We shall use a basic tool in 3-manifold theory, called band sums of disks, to prove Theorem 1. The argument in this paper is self-contained. We shall give an outline of the proof of Theorem 1 in Chapter 2.

## 2 The outline of Theorem 1

In this chapter, we shall introduce the ideas of the proof of Theorem 1. Before doing this, we first give some notations and related basic observations.

### 2.1 The surface generated by an abstract tree

**Definition 2.1.1.** (1) Let  $I = [-1, 1]$ .

(2) Suppose that  $a$  is an arc in a surface  $F$ ,  $a \times I$  be a neighborhood of  $a$  in  $F$  such that  $a = a \times \{0\}$ . For a sub-arc  $b$  of  $a$ , we denote  $b \times I \subset a \times I$  by  $(b \times I)_a$ . In this case,  $b \times I$  and  $a \times I$  have the same width.

Let  $m(l) = \{m^1, \dots, m^l\}$  be a subset of  $\{1, \dots, n\}$  possibly not in a natural order. This means that it is possible  $m^i > m^j$  when  $i < j$ .

**Definition 2.1.2.** Suppose that  $F$  is a closed surface,  $P_0, \dots, P_k$  are pairwise disjoint disks in  $F$ , and  $e_{m^1}, \dots, e_{m^l}$  are pairwise disjoint arcs in  $F$  satisfying the following conditions:

(1)  $\partial e_\gamma \subset \cup_{f=0}^k \partial P_f$ .

(2) If we denote by  $e'_\gamma$  the arc obtained by pushing  $inte_\gamma$  off  $\cup_f P_f$  in  $F \times I$ , then each component of  $\cup_f P_f \cup_\gamma e'_\gamma$  is a tree when we take  $P_f$  as a fat vertex and  $e'_\gamma$  as an edge. Then we say  $\cup_f P_f \cup_\gamma e_\gamma$  is an abstract tree.

Let  $\cup_f P_f \cup_\gamma e_\gamma$  be an abstract tree in a closed surface, and  $e_\gamma \times I$  be a regular neighborhood of  $e_\gamma$  in  $F$  satisfying the conditions:

(1\*) If  $\lambda > \gamma \in m(l)$ , then either  $e_\lambda \cap e_\gamma \times I = \emptyset$  or each component of  $e_\lambda \cap e_\gamma \times I$  is an arc  $c \subset inte_\lambda$  which is a core of  $e_\gamma \times (0, 1)$ , and each component of  $e_\lambda \times I \cap e_\gamma \times I$  is  $(c \times I)_\lambda \subset e_\gamma \times (0, 1)$ .

(2\*) Each component of  $inte_\lambda \times I \cap (\cup_f P_f \cup_{\gamma < \lambda} e_\gamma \times I)$  is  $(b \times I)_\lambda$  where  $b \subset inte_\gamma$  is a properly embedded arc in  $\cup_f P_f \cup_{\gamma < \lambda} e_\gamma \times I$ .

(3\*) For  $\gamma \in m(l)$ ,  $(\partial e_\gamma) \times I \subset \cup_f \partial P_f$ , for  $\lambda \neq \gamma \in m(l)$ ,  $(\partial e_\gamma) \times I \cap (\partial e_\lambda) \times I = \emptyset$ .

**Lemma 2.1.3.** If Conditions (1\*), (2\*) and (3\*) are satisfied, then  $S = \cup_f P_f \cup_{\gamma \in m(l)} e_\gamma \times I$  is a compact surface.

**Proof.** Let  $\gamma = \text{Min}m(l)$ . Then, by Condition (2\*), each component of  $\text{inte}_\gamma \times I \cap (\cup_f P_f)$  is  $(b \times I)_\gamma$  where  $b \subset \text{inte}_\gamma$  is a properly embedded arc in  $\cup_f P_f$ . Hence  $b$  is compact and  $\partial b \cap \partial e_\gamma = \emptyset$ . We denote by  $b_1, \dots, b_\alpha$  the components of  $e_\gamma - \cup_f \text{int} P_f$ . Then  $b_i$  is an arc with  $\partial b_i \subset \cup_f \partial P_f$ . Hence  $\cup_f P_f \cup e_\gamma \times I = \cup_f P_f \cup \cup_{i=1}^\alpha (b_i \times I)_\gamma$  is a surface. By Conditions (1\*), (2\*) and (3\*) and induction on  $m(l) = \{m^1, \dots, m^l\}$ , we can prove Lemma 2.1.3. Q.E.D.

**Definition 2.1.4.** Let  $\cup_f P_f \cup_\gamma e_\gamma$  be an abstract tree in a closed surface. If Conditions (1\*), (2\*) and (3\*) are satisfied, then  $S = \cup_f P_f \cup_{\gamma \in m(l)} e_\gamma \times I$  is called a surface generated by  $\cup_f P_f \cup_\gamma e_\gamma$ .

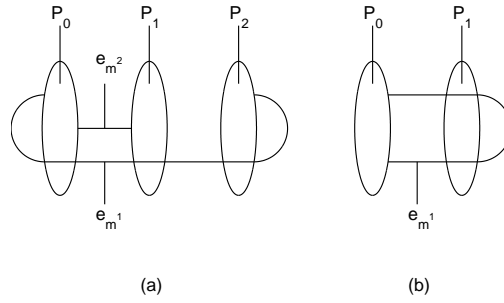


Figure 1

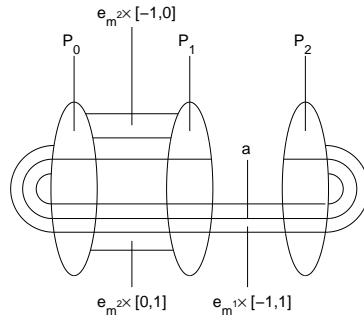


Figure 2

**Example 1:**

(1). Suppose that  $P_0, P_1, P_2$  are three disks in a surface  $F$  and  $e_{m^1}, e_{m^2}$  are two arcs in  $F$  as in Figure 1(a). Then  $\cup_f P_f \cup_\gamma e_\gamma \times I$  is an abstract tree.

(2). Suppose that  $P_0, P_1$  and  $e_{m^1}$  are as in Figure 1(b). Then  $P_1 \cup P_2 \cup e_{m^1}$  is not an abstract tree. In this case, if  $e'_{m^1}$  is an arc such that  $\partial e_{m^1} = \partial e'_{m^1}$  and  $\text{inte}'_{m^1}$  is disjoint from  $P_0 \cup P_1$ , then  $P_0 \cup P_1 \cup e'_{m^1}$  is not a tree.

(3). Figure 2 is a surface generated by an abstract tree  $\cup_f P_f \cup_\gamma e_\gamma$  as in Figure 1(a). In this case,  $m^1 > m^2$ . Note that the surface generated by a fixed abstract tree is not unique.

Now we consider a kind of special arcs in a surface generated by an abstract tree.

**Definition 2.1.5.** Let  $S$  be a surface generated by an abstract tree  $\cup_f P_f \cup_\gamma e_\gamma$ . If  $a$  is a properly embedded arc in  $S$  such that

- 1) for  $\gamma \in m(l)$ , each component of  $a \cap e_\gamma \times I$  is a core of  $e_\gamma \times (0, 1)$  which is contained in  $\text{inta}$ ,
- 2) each component of  $a \cap (\cup_\gamma e_\gamma \times I)$  is a core of  $e_\lambda \times (0, 1)$  for some  $\lambda \in m(l)$ ,
- 3) for each  $\lambda \in m(l)$ , there is at most one component of  $a \cap (\cup_\gamma e_\gamma \times I)$  which is a core of  $e_\lambda \times (0, 1)$ .

Then  $a$  is said to be regular in  $S$ .

**Example 2:**

The arc  $a$  in Figure 2 is regular in  $P_0 \cup P_1 \cup P_2 \cup e_{m^1} \times I \cup e_{m^2} \times I$ . In this case,  $a$  intersects  $e_{m^2} \times I$  in two cores of  $e_{m^2} \times (0, 1)$ ,  $e_{m^1} \times I$  in one core of  $e_{m^1} \times (0, 1)$ ,  $e_{m^1} \times I \cup e_{m^2} \times I$  in two components, one of which is a core of  $e_{m^1} \times (0, 1)$  and the other is a core of  $e_{m^2} \times (0, 1)$ .  $a - (\text{inte}_{m^1}) \times I \cup (\text{inte}_{m^2} \times I)$  contains three components  $a_0, a_1, a_2$  with  $a_i \subset P_i$ .

**Lemma 2.1.6.** Let  $a$  be a regular arc in a surface  $S$  generated by an abstract tree  $\cup_f P_f \cup_\gamma e_\gamma$ . Then  $a = \cup_{i=1}^{\theta(a)} a_{f_i} \cup_{i=1}^{\theta(a)-1} e_{\gamma_i, a}$  satisfying the following conditions:

- 1)  $0 \leq f_i \leq k$ ,  $a_{f_i}$  is a properly embedded arc in  $P_{f_i}$  which is disjoint from  $\cup_{\gamma \in m(l)} \text{inte}_\gamma \times I$ .
- 2)  $\gamma_i \in m(l)$ , and  $e_{\gamma_i, a}$  is a core of  $e_{\gamma_i} \times (0, 1)$ .
- 3) For  $1 \leq i \leq \theta(a) - 1$ ,  $\partial_2 a_{f_i} = \partial_1 e_{\gamma_i, a}$  and for  $2 \leq i \leq \theta(a) - 1$ ,  $\partial_1 a_{f_{i+1}} = \partial_2 e_{\gamma_i, a}$ ,  $\partial_1 a = \partial_1 a_{f_1}$ ,  $\partial_2 a = \partial_2 a_{f_{\theta(a)}}$ .
- 4) For  $1 \leq i \neq r \leq \theta(a)$ ,  $f_i \neq f_r$ , for  $1 \leq i \neq r \leq \theta(a) - 1$ ,  $\gamma_i \neq \gamma_r$ .
- 5) For each  $f$ ,  $P_f \cap (a - \cup_\gamma \text{inte}_\gamma \times I)$  contains at most one component.

**Proof.** Since  $a$  is regular in  $S$ , By Definition 2.1.5(1) and (2),  $\partial a$  is disjoint from  $\cup_{\gamma} e_{\gamma} \times I$ , and each component of  $a \cap (\cup_{\gamma} e_{\gamma} \times I)$  is a core of  $e_{\lambda} \times (0, 1)$  for some  $\lambda \in m(l)$ . By Definition 2.1.4,  $(\partial e_{\gamma}) \times I \subset \cup_f \partial P_f$ , each component of  $a - \cup_{\gamma} \text{inte}_{\gamma} \times I$  is an properly embedded arc in  $P_f$  for some  $0 \leq f \leq k$ . Hence  $a = \cup_{j=1}^{\theta(a)} a_{f_j} \cup_{j=1}^{\theta(a)-1} e_{\gamma_j, a}$ , and (1), (2), (3) holds.

By Definition 2.1.5(3),  $\gamma_i \neq \gamma_r$  for  $1 \leq i \neq r \leq \theta(a) - 1$ . Now if  $f_i = f_r$  for  $1 \leq i \neq r \leq \theta(a)$ , then  $\cup_f P_f \cup_{\gamma} e_{\gamma}$  is not an abstract tree, a contradiction. Hence (4) holds. (5) follows from (1) and (4). Q.E.D.

**Remark 2.1.7.** The properties of regular arcs in Lemma 2.1.6 are important in the proof of Theorem 1. Lemmas 3.1.2 and 3.3.1 follow from Lemma 2.1.6 and they take roles in the inductive proof of Theorem 1.

**Lemma 2.1.8.** Let  $S = \cup_f P_f \cup_{\gamma \in m(l)} e_{\gamma} \times I$  be a surface generated by an abstract tree. Then  $S_j = \cup_f P_f \cup_{\gamma < j} e_{\gamma} \times I$  is also a surface generated by an abstract tree where  $j$  is any integer.

**Proof.** Since  $\cup_f P_f \cup_{\gamma \in m(l)} e_{\gamma}$  is an abstract tree,  $\cup_f P_f \cup_{\gamma < j} e_{\gamma}$  is also an abstract tree. Obviously, Conditions (1\*), (2\*) and (3\*) are satisfied. Hence  $S_j = \cup_f P_f \cup_{\gamma < j} e_{\gamma} \times I$  is a surface generated by  $\cup_f P_f \cup_{\gamma < j} e_{\gamma}$ . Q.E.D.

## 2.2 The element of the induction

It is easy to see that if one of  $\mathcal{V}_+ \cup \mathcal{W}_+$  and  $\mathcal{V}_- \cup \mathcal{W}_-$  is stabilized then  $\mathcal{V} \cup \mathcal{W}$  is stabilized. So in order to obtain a contradiction, we may assume that each of  $\mathcal{V}_+ \cup \mathcal{W}_+$  and  $\mathcal{V}_- \cup \mathcal{W}_-$  is unstabilized and  $\mathcal{V} \cup \mathcal{W}$  is stabilized.

**Assumption(\*).** (1) Let  $(V, W)$  be a pair of stabilized disks such that  $V \subset \mathcal{V}$ ,  $W \subset \mathcal{W}$ .

Now  $V$  intersects  $W$  in only one point  $x$ . Recalling the two disks  $B_{\mathcal{V}}$  and  $B_{\mathcal{W}}$  defined in Chapter 1. We may assume that each component of  $V \cap B_{\mathcal{V}}$  is an arc in both  $V$  and  $B_{\mathcal{V}}$ , each component of  $W \cap B_{\mathcal{W}}$  is an arc in both  $W$  and  $B_{\mathcal{W}}$ . It is easy to see that if  $V \cap B_{\mathcal{V}} = \emptyset$  or  $W \cap B_{\mathcal{W}} = \emptyset$ , then one of  $\mathcal{V}_+ \cup \mathcal{W}_+$  and  $\mathcal{V}_- \cup \mathcal{W}_-$  is stabilized. So we may assume that  $V \cap B_{\mathcal{V}} \neq \emptyset$  and  $W \cap B_{\mathcal{W}} \neq \emptyset$ .

**Assumption (\*)**. (2)  $x \in \partial_+ \mathcal{V}_+ - B_{\mathcal{V}}$ .

(3)  $|V \cap B_{\mathcal{V}}| = m$  and  $|W \cap B_{\mathcal{W}}| = n$ .

Now consider  $V \cap B_{\mathcal{V}}$ . By assumption, each component  $e$  of  $V \cap B_{\mathcal{V}}$  separates  $V$  into two disks  $V'_e$  and  $V''_e$  such that  $x \in \partial V'_e$ . Now we denote by  $V_e$  the disk in  $V''_e$  which is bounded by  $e$ , with some components in  $V \cap B_{\mathcal{V}}$  and some arcs in  $\partial V$ , such that  $\text{int} V_e$  is disjoint from  $B_{\mathcal{V}}$  as in Figure 3. Then  $V_e$  is a properly embedded disk in  $\mathcal{V}_+$  or  $\mathcal{V}_-$ .

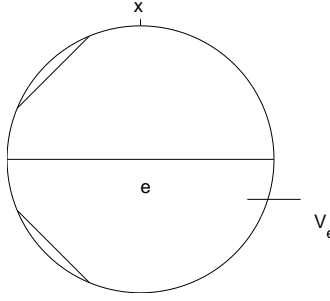


Figure 3

**Definition 2.2.1.** A component  $e$  in  $V \cap B_{\mathcal{V}}$  is labeled a symbol  $s(e)$  where  $s(e) = +$  if  $V_e \subset \mathcal{V}_+$  and  $s(e) = -$  if  $V_e \subset \mathcal{V}_-$ .

Similarly, each component  $e$  of  $W \cap B_{\mathcal{W}}$  separates  $W$  into two disks  $W'_e$  and  $W''_e$  such that  $x \in \partial W'_e$ . Now we denote by  $W_e$  the disk in  $W''_e$  which is bounded by  $e$  with some components in  $W \cap B_{\mathcal{W}}$  and some arcs in  $\partial W$  such that  $\text{int} W_e$  is disjoint from  $B_{\mathcal{W}}$ . Then  $W_e$  is a properly embedded disk in  $\mathcal{W}_+^*$  or  $\mathcal{W}_-^*$  defined in Section 1.

**Definition 2.2.2.** A component  $e$  in  $W \cap B_{\mathcal{W}}$  is labeled a symbol  $s(e)$  where  $s(e) = +$  if  $W_e \subset \mathcal{W}_+^*$  and  $s(e) = -$  if  $W_e \subset \mathcal{W}_-^*$ .

By Assumption(\*), we number the components of  $V \cap B_{\mathcal{V}}$ ,  $v_1, \dots, v_m$ , and the components of  $W \cap B_{\mathcal{W}}$ ,  $w_1, \dots, w_n$ , so that if  $V''_{v_i} \subset V''_{v_k}$  and  $W''_{w_j} \subset W''_{w_l}$ , then  $i < k$ ,  $j < l$ . Now we denote by  $V_i$  the disk  $V_{v_i}$ ,  $W_j$  the disk  $W_{w_j}$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .

**Definition 2.2.3.** For each  $v_i$  in  $V \cap B_{\mathcal{V}}$  and each  $w_j$  in  $W \cap B_{\mathcal{W}}$ , let  $I(v_i) = \{r \mid v_r \neq v_i \subset \partial V_i\}$ , and  $I(w_j) = \{r \mid w_r \neq w_j \subset \partial W_j\}$ .

The following two lemmas are immediately from definitions.



**Lemma 2.2.4.** 1) If  $r \in I(v_i)$ , then  $r < i$ , and

2) if  $r \in I(w_j)$ , then  $r < j$ .

**Proof.** We need only to prove (1).

If  $r \in I(v_i)$ , then  $V_r'' \subset V_i''$ . By definitions,  $r < i$ .

**Lemma 2.2.5.** (1)  $s(v_i) = +$  if and only if  $s(v_r) = -$  for each  $r \in I(v_i)$ .

(2) If  $r \in I(v_i), I(v_k)$ , then  $k = i$ .

(3)  $s(w_j) = +$  if and only if  $s(w_r) = -$  for each  $r \in I(w_j)$ .

(4) If  $r \in I(w_j), I(w_k)$ , then  $k = j$ .

**Proof.** This lemma is immediately from definitions and simple observations. Q.E.D.

Since  $x \in \partial_+ \mathcal{V}_+$ , each component of  $V \cap \mathcal{V}_+$  is either the disk  $V_i$  for some  $1 \leq i \leq m$  with  $s(v_i) = +$  or a disk containing  $x$ , denoted by  $V_x$ . By definition of  $V_i$ ,  $V_x$  is disjoint from  $V_i$  for  $s(v_i) = +$ . Now each component of  $V \cap \mathcal{V}_-$  is the disk  $V_i$  for some  $1 \leq i \leq m$  with  $s(v_i) = -$ . Similarly, each component of  $W \cap \mathcal{W}_+^*$  is either the disk  $W_j$  for some  $1 \leq j \leq n$  with  $s(w_j) = +$  or a disk containing  $x$ , denoted by  $W_x$ , each component of  $W \cap \mathcal{W}_-^*$  is the disk  $W_j$  for some  $1 \leq j \leq n$  with  $s(w_j) = -$ .

**Definition 2.2.6.** Let  $I(v) = \{r \mid v_r \in V_x \cap B_{\mathcal{V}}\}$  and  $I(w) = \{r \mid w_r \in W_x \cap B_{\mathcal{W}}\}$ .

**Lemma 2.2.7.** 1) If  $r \in I(v)$ , then  $s(v_r) = -$ .

2) If  $r \in I(w)$ , then  $s(w_r) = -$ .

3)  $m \in I(v), n \in I(w)$ .

**Proof.** By assumption,  $x \in \partial_+ \mathcal{V}_+$ . Hence  $V_x \subset \mathcal{V}_+$ ,  $W_x \subset \mathcal{W}_+$ . If  $r \in I(v)$ , then  $V_r \subset \mathcal{V}_-$ . If  $r \in I(w)$ , then  $W_r \subset \mathcal{W}_-$ .

Now  $v_m$  separates  $V$  into  $V_{v_m}'$  and  $V_{v_m}''$  such that  $x \in V_{v_m}'$ . Suppose, otherwise,  $m \notin I(v)$ . By definition,  $v_m$  is disjoint from  $\partial V_x$ . This means that there is an integer  $1 \leq i < m$  such that  $v_i$  separates  $v_m$  and the point  $x$  in  $V_{v_m}'$ . By definition,  $V_{v_m}'' \subset V_{v_i}''$ . Hence  $m < i$ , a contradiction. Q.E.D.

Recalling the definitions of  $\mathcal{V}_+ \cup \mathcal{W}_+$  and  $\mathcal{V}_- \cup \mathcal{W}_-$  in section 1. Now  $W_j$  and  $W_x$  are properly embedded in  $\mathcal{W}_+^*$  or  $\mathcal{W}_-^*$  for  $1 \leq j \leq n$ . It is easy to see that for each  $j$ , there is an arc  $w_{j,v}$  in  $B_{\mathcal{V}}$  such that  $w_{j,v} \cup w_j$  bounds a disk in  $H_+^3$ , denoted by  $W_{j,+}$ , and a disk

in  $H^3$ , denoted by  $W_{j,-}$  as in Figure 4. Note that for each  $j \neq k$ ,  $W_{j,+} \cap W_{k,+} = \emptyset$  and  $W_{j,-} \cap W_{k,-} = \emptyset$ . Thus if  $s(w_j) = +$ , then  $W_{j,v} = W_j \cup W_{j,+} \cup_{r \in I(w_j)} W_{r,+}$  is a properly embedded disk in  $\mathcal{W}_+$ , and if  $s(w_j) = -$ , then  $W_{j,v} = W_j \cup W_{j,-} \cup_{r \in I(w_j)} W_{r,-}$  is a properly embedded disk in  $\mathcal{W}_-$ . Specially,  $W_{x,v} = W_x \cup_{r \in I(w)} W_{r,+}$  is a properly embedded disk in  $\mathcal{W}_+$ .

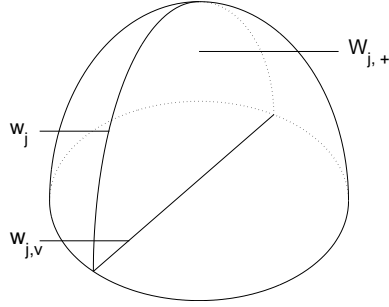


Figure 4

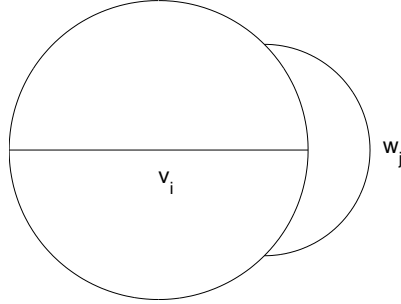


Figure 5

**Lemma 2.2.8.**  $\{v_1, \dots, v_m\}$  and  $\{w_{1,v}, \dots, w_{n,v}\}$  are two sets of pairwise disjoint arcs properly embedded in  $B_{\mathcal{V}}$  such that

- (1)  $v_i$  intersects  $w_{j,v}$  in at most one point, and
- (2)  $w_{j,v} \cap \partial v_i = \emptyset$  for each  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .

**Proof.** Since  $v_i \subset V \cap B_{\mathcal{V}}$ ,  $w_j \subset W \cap B_{\mathcal{W}}$  and  $W \cap V = \{x\}$ , By Assumption(\*),  $v_i$  is properly embedded in  $B_{\mathcal{V}}$  and  $w_j$  is properly embedded in  $B_{\mathcal{W}}$  such that  $v_i$  is disjoint from  $w_j$ . Note that  $\partial B_{\mathcal{V}} = \partial B_{\mathcal{W}}$ . Hence either the two end points of  $w_j$  lie in the same component of  $\partial B_{\mathcal{V}} - \partial v_i$  or the two end points of  $w_j$  lie in the distinct components of  $\partial B_{\mathcal{V}} - \partial v_i$  as in Figure 5. Hence the lemma holds. Q.E.D.

**Lemma 2.2.9.** There are four sets of pairwise disjoint properly embedded disks  $\{V_i \mid s(v_i) = +\} \cup \{V_x\}$  in  $\mathcal{V}_+$ ,  $\{V_i \mid s(v_i) = -\}$  in  $\mathcal{V}_-$ ,  $\{W_{j,v} \mid s(w_j) = +\} \cup \{W_{x,v}\}$  in  $\mathcal{W}_+$ , and  $\{W_{j,v} \mid s(w_j) = -\}$  in  $\mathcal{W}_-$  satisfying the following conditions:

(1)  $V_i \cap B_{\mathcal{V}} = v_i \cup_{r \in I(v_i)} v_r$ ,  $W_{j,v} \cap B_{\mathcal{V}} = w_{j,v} \cup_{r \in I(w_j)} w_{r,v}$ ,  $V_x \cap B_{\mathcal{V}} = \cup_{r \in I(v)} v_r$ ,  $W_{x,v} \cap B_{\mathcal{V}} = \cup_{r \in I(w)} w_{r,v}$ .

(2) If  $s(v_i) = +$ ,  $s(w_j) = +$ , then  $V_i \cap W_{j,v} = V_i \cap W_{j,v} \cap B_{\mathcal{V}}$ ,  $V_i \cap W_{x,v} = V_i \cap W_{x,v} \cap B_{\mathcal{V}}$ ,  $V_x \cap W_j = V_x \cap W_j \cap B_{\mathcal{V}}$ ,  $V_x \cap W_{x,v} = (V_x \cap W_{x,v} \cap B_{\mathcal{V}}) \cup \{x\}$ .

(3) If  $s(v_i) = -$ ,  $s(w_j) = -$ , then  $V_i \cap W_{j,v} = V_i \cap W_{j,v} \cap B_{\mathcal{V}}$ .

**Proof.** (1) follows from the construction of  $w_{j,v}$ .

Since  $W \cap V = W_x \cap V_x = \{x\}$ ,  $W_j \cap V_i = \emptyset$  for each  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . By the constructions of  $W_{j,v}$ ,  $W_{x,v}$ ,  $w_{j,v}$ , (2) and (3) holds. Q.E.D.

## 2.3 Outline of the proof of Theorem 1

### The idea of the Proof of Theorem 1

Now let  $\mathcal{V}_+^1 = \partial_+ \mathcal{V}_+ \times I \cup_{s(v_i)=+} N(V_i) \cup N(V_x) \cup \{3 - handles\}$  where  $N(V_i)$  and  $N(V_x)$  are regular neighborhoods of  $V_i$  and  $V_x$  in  $\mathcal{V}_+$ ,  $\mathcal{V}_-^1 = \partial_+ \mathcal{V}_- \times I \cup_{s(v_i)=-} N(V_i) \cup \{3 - handles\}$  where  $N(V_i)$  is a regular neighborhood of  $V_i$  in  $\mathcal{V}_-$ . Then  $\mathcal{V}_+^1 \subset V_+$  and  $\mathcal{V}_-^1 \subset V_-$  are two compression bodies. Let  $\mathcal{W}_+^1 = \partial_+ \mathcal{W}_+ \times I \cup_{s(w_j)=+} N(W_{j,v}) \cup N(W_{x,v}) \cup \{3 - handles\}$ , where  $N(W_{j,v})$  and  $N(W_{x,v})$  are regular neighborhoods of  $W_{j,v}$  and  $W_{x,v}$  in  $\mathcal{W}_+$ ,  $\mathcal{W}_-^1 = \partial_+ \mathcal{W}_- \times I \cup_{s(w_j)=-} N(W_{j,v}) \cup \{3 - handles\}$  where  $N(W_{j,v})$  is a regular neighborhood of  $W_{j,v}$  in  $\mathcal{W}_-$ .

Now  $W$  and  $V$  defined in Section 2.2 are a pair of stabilized disks of the connected sum of  $\mathcal{V}_+^1 \cup \mathcal{W}_+^1$  and  $\mathcal{V}_-^1 \cup \mathcal{W}_-^1$ . If Theorem 1 is true, then one of  $\mathcal{V}_+^1 \cup \mathcal{W}_+^1$  and  $\mathcal{V}_-^1 \cup \mathcal{W}_-^1$ , say  $\mathcal{V}_+^1 \cup \mathcal{W}_+^1$ , is stabilized. This means that there are two essential disks  $W' \subset \mathcal{W}_+^1$  and  $V' \subset \mathcal{V}_+^1$  such that  $W'$  intersects  $V'$  in only one point. Hence  $W'$  is obtained by doing band sums from  $W_{j,v}$  with  $s(w_j) = +$  and  $W_{x,v}$ , and  $V'$  is obtained by doing band sums from  $V_i$  with  $s(v_i) = +$  and  $V_x$ . We do want to do this.

Recalling  $I(v_i), I(w_j), I(v), I(w)$ ,  $s(v_i), s(w_j)$  which are defined in Section 2.2.

**Definition 2.3.1.** For  $0 \leq k \leq m$  and  $m(k) = \{m^0, \dots, m^k\} \subset \{1, \dots, n\}$ . Let  $I(v_i, k) = I(v_i) - \{1, \dots, k\}$ ,  $I(w_j, k) = I(w_j) - m(k)$ ,  $I(v, k) = I(v) - \{1, \dots, k\}$  and  $I(w, k) = I(w) - m(k)$ .

By definition, if  $k > l$ , then  $I(v_i, k) \subset I(v_i, l)$ ,  $I(v, k) \subset I(v, l)$ . Furthermore,  $I(v_i, m) = I(v, m) = \emptyset$ . By definitions,  $I(v_i), I(v) \subset \{1, \dots, m\}$ . Hence  $I(v_i, k) = I(v_i) - \{1, \dots, k\}$ ,  $I(v, k) = I(v) - \{1, \dots, k\}$ . Similarly,  $I(w_j), I(w) \subset \{1, \dots, n\}$ .

To prove Theorem 1, we only need to prove the following propositions:

**Proposition 1.** For each  $0 \leq k \leq m$ , there are a surface  $P^k$  in  $\partial_+ \mathcal{V}_+$ , a subset of  $\{1, \dots, n\}$ , say  $m(k) = \{m^0, m^1, \dots, m^k\}$ , and two sets of pairwise disjoint arcs  $\{v_i^k \mid k+1 \leq i \leq m\}$  and  $\{w_j^k \mid j \in \{1, \dots, n\} - m(k)\}$  in  $P^k$  satisfying the following conditions:

(1)  $P^k = \cup_{f=0}^k D_f^k \cup_{\gamma \in m(k)} b_\gamma^k \times I$  is a surface generated by an abstract tree  $\cup_{f=0}^k D_f^k \cup_{\gamma \in m(k)} b_\gamma^k$  such that

(i) for each  $\lambda \subset m(k)$  with  $s(w_\lambda) = +$ ,  $(\text{int} b_\lambda^k \times I) \cap (\cup_{\gamma < \lambda} b_\gamma^k \times I \cup_f D_f^k) = \cup_{r \in I(w_\lambda, k)} (w_r^k \times I)_\lambda$ ;

(ii) for each  $\lambda \subset m(k)$  with  $s(w_\lambda) = -$ ,  $\text{int} b_\lambda^k \times I$  is disjoint from  $\cup_{f=0}^k D_f^k \cup_{\gamma < \lambda} b_\gamma^k \times I$ .

(2) For  $j \notin m(k)$ ,  $w_j^k$  is regular in  $\cup_{\gamma < j} b_\gamma^k \times I \cup_f D_f^k$ .

(3) For  $j \notin m(k)$  and  $\gamma \in m(k)$ , if  $j < \gamma$ , then either  $j \in I(w_\gamma, k)$  with  $s(w_\gamma) = +$  or  $w_j^k$  is disjoint from  $b_\gamma^k \times I$ .

(4) For each  $i \geq k+1$ ,  $v_i^k$  is a properly embedded arc in  $P^k$  lying in  $D_f^k$  for some  $f$ . Furthermore, for each  $j \in \{1, \dots, n\} - m(k)$ ,  $w_j^k - \cup_{\gamma < j} \text{int} b_\gamma^k \times I$  intersects  $v_i^k$  in at most one point.

(5) For each  $i \geq k+1$ ,  $j \in \{1, \dots, n\} - m(k)$  and  $\gamma \in m(k)$ ,  $\partial v_i^k \cap w_j^k = \emptyset$ ,  $\partial v_i^k \cap b_\gamma^k \times I = \emptyset$ .

**Definition 2.3.2.** If  $w_j^k - \cup_{\gamma < j} \text{int} b_\gamma^k \times I$  intersects  $v_i^k$  in one point, then we say  $i \in L(w_j^k)$  and  $j \in L(v_i^k)$ .

**Proposition 2.** For each  $1 \leq k \leq m$  and  $j \in \{1, \dots, n\} - m(k)$ .

(1) If  $j \notin L(v_k^{k-1})$ , then  $L(w_j^k) = L(w_j^{k-1})$ .

(2) If  $j \in L(v_k^{k-1})$ , then  $L(w_j^k) = L(w_j^{k-1}) \cup L(w_m^{k-1}) - L(w_j^{k-1}) \cap L(w_m^{k-1})$ .

(3)  $m^0 = \emptyset$  and  $m^k = \text{Min} L(c_k^{k-1})$ .

Where  $m(k) = \{m^0, \dots, m^k\}$  is as in Proposition 1.

**Proposition 3.** For each  $0 \leq k \leq m$ , there are two sets of pairwise disjoint disks  $\{V_i^k \mid k+1 \leq i \leq m \text{ with } s(v_i) = +\} \cup \{V_x^k\}$  properly embedded in  $\mathcal{V}_+$  and  $\{W_j^k \mid j \in \{1, \dots, n\} - m(k) \text{ with } s(w_j) = +\} \cup \{W_x^k\}$  properly embedded in  $\mathcal{W}_+$  satisfying the following conditions:

- (1)  $V_i^k \cap P^k = v_i^k \cup_{r \in I(v_i, k)} v_r^k$ ,  $W_j^k \cap P^k = w_j^k \cup_{r \in I(w_j, k)} w_r^k$ ,  $V_x^k \cap P^k = \cup_{r \in I(v, k)} v_r^k$ ,  $W_x^k \cap P^k = \cup_{r \in I(w, k)} w_r^k$ .
- (2)  $V_i^k \cap W_j^k = V_i^k \cap W_j^k \cap P^k$ ,  $V_i^k \cap W_x^k = V_i^k \cap W_x^k \cap P^k$ ,  $V_x^k \cap W_j^k = V_x^k \cap W_j^k \cap P^k$ ,  $V_x^k \cap W_x^k = (V_x^k \cap W_x^k \cap P^k) \cup \{x\}$  where  $x \in W_{x, v} \cup V_x$  in Lemma 2.2.9.

Recalling the set  $\{m^0, m^1, \dots, m^k\}$  in Proposition 1.

**Proposition 4.** For each  $1 \leq k \leq m$ , there are a surface  $F^k$  in  $\partial_+ \mathcal{V}_-$ , and two sets of pairwise disjoint arcs  $\{c_i^k \mid k+1 \leq i \leq m\}$  and  $\{d_j^k \mid j \in \{1, \dots, n\} - m(k)\}$  in  $F^k$  satisfying the following conditions:

- (1)  $F^k = \cup_{f=0}^k E_f^k \cup_{\gamma \in m(k)} e_\gamma^k \times I$  is a surface generated by an abstract tree  $\cup_{f=0}^k E_f^k \cup_{\gamma \in m(k)} e_\gamma^k$  such that
  - (i) for each  $\lambda \subset m(k)$ , if  $s(w_\lambda) = -$ , then  $(\text{int} e_\lambda^k \times I) \cap (\cup_{\gamma < \lambda} e_\gamma^k \times I \cup_f E_f^k) = \cup_{r \in I(w_\lambda, k)} (d_r^k \times I)_\lambda$ ;
  - (ii) if  $s(w_\lambda) = +$ ,  $\text{int} e_\lambda^k \times I$  is disjoint from  $\cup_{f=0}^k E_f^k \cup_{\gamma < \lambda} e_\gamma^k \times I$ .
- (2) For  $j \notin m(k)$ ,  $d_j^k$  is regular in  $\cup_{\gamma < j} e_\gamma^k \times I \cup_f E_f^k$ .
- (3) For  $j \notin m(k)$  and  $\gamma \in m(k)$ , if  $j < \gamma$ , then either  $j \in I(w_\gamma, k)$  with  $s(w_\gamma) = -$  or  $d_j^k$  is disjoint from  $e_\gamma^k \times I$ .
- (4) For each  $i \geq k+1$ ,  $c_i^k$  is a properly embedded arc in  $F^k$  lying in  $E_f^k$  for some  $f$ . Furthermore, for each  $j \in \{1, \dots, n\} - m(k)$ ,  $d_j^k - \cup_{\gamma < j} \text{int} e_\gamma^k \times I$  intersects  $c_i^k$  in at most one point.
- (5) For each  $i \geq k+1$ ,  $j \in \{1, \dots, n\} - m(k)$  and  $\gamma \in m(k)$ ,  $\partial c_i^k \cap d_j^k = \emptyset$ ,  $\partial c_i^k \cap e_\gamma^k \times I = \emptyset$ .

**Definition 2.3.3.** If  $d_j^k - \cup_{\gamma < j} \text{int} e_\gamma^k \times I$  intersects  $c_i^k$  in one point, then we say  $i \in L(d_j^k)$  and  $j \in L(c_i^k)$ .

**Proposition 5.**  $L(c_i^0) = L(v_i^0)$ ,  $L(d_j^0) = L(w_j^0)$ . For each  $1 \leq k \leq m$  and  $j \notin m(k)$ .

(1) If  $j \notin L(c_k^{k-1})$ , then  $L(d_j^k) = L(d_j^{k-1})$ .

(2) If  $j \in L(c_k^{k-1})$ , then  $L(d_j^k) = L(d_j^{k-1}) \cup L(d_{m^k}^{k-1}) - L(d_j^{k-1}) \cap L(d_{m^k}^{k-1})$ .

**Proposition 6.** For each  $0 \leq k \leq m$ , there are two sets of pairwise disjoint disks  $\{V_i^k \mid k+1 \leq i \leq m \text{ with } s(v_i) = -\}$  properly embedded in  $\mathcal{V}_-$  and  $\{W_j^k \mid j \in \{1, \dots, n\} - m(k) \text{ with } s(w_j) = -\}$  properly embedded in  $\mathcal{W}_-$  such that

(1)  $V_i^k \cap F^k = c_i^k \cup_{r \in I(v_i, k)} c_r^k$ ,  $W_j^k \cap F^k = d_j^k \cup_{r \in I(w_j, k)} d_r^k$ ;

(2)  $V_i^k \cap W_j^k = V_i^k \cap W_j^k \cap F^k$ .

Now we prove Theorem 1 under the assumption that Propositions 1-6 are true.

**The proof of Theorem 1.** Suppose that  $k = m$ . By the definition of  $I(v, k)$ ,  $I(v, m) = I(v) - \{1, \dots, m\} = \emptyset$ . By Proposition 3(2),  $V_x^m \cap W_x^m = (V_x^m \cap W_x^m \cap P^m) \cup \{x\}$ . By Proposition 3(1),  $V_x^m \cap P^m = \cup_{r \in I(v, m)} v_r^m$ ,  $W_x^m \cap P^m = \cup_{r \in I(w, m)} w_r^m$ . Hence  $V_x^m \cap W_x^m \cap P^m = \emptyset$  and  $V_x^m \cap W_x^m = \{x\}$ . This means that  $\mathcal{V}_+ \cup \mathcal{W}_+$  is stabilized. Q.E.D.

**Remark.** Though Theorem 1 follows immediately from Proposition 3. But Propositions 1, 2, 4, 5 and 6 are necessary in the inductive proof of Proposition 3. See Section 2.5

## 2.4 The organizing of the inductive proofs of Propositions 1-6

We organize the proofs of Propositions 1-6 as follows:

In Section 2.5, we shall prove Propositions 1-6 for  $k = 0$ . Furthermore, we shall introduce the ideas of the inductive proofs of Propositions 1-6.

In Chapter 3, we shall study properties of  $\{v_i^k, w_j^k, b_\gamma^k \times I, P^k\}$  and  $\{c_i^k, d_j^k, e_\gamma^k \times I, F^k\}$  under the assumptions that Propositions 1-6 are true for  $k \leq l$ .

In Chapter 4, we shall prove that Propositions 4-6 are true for  $k = l + 1$  under the assumptions:  $s(v_{l+1}) = +$  and Propositions 1-6 are true for  $k \leq l$ .

In Chapter 5, we shall prove that Propositions 4-6 are true for  $k = l + 1$  under the assumptions:  $s(v_{l+1}) = -$  and Propositions 1-6 are true for  $k \leq l$ .

In Chapter 6, we shall finish the proofs of Propositions 1-6.

## 2.5 The proofs of Propositions 1-6 for $k = 0$

**Lemma 2.5.1.** Propositions 1-3 are true for  $k = 0$ .

**Proof.** By the definition,  $B_V$  is a disk in  $\partial_+ \mathcal{V}_+$ . Hence  $B_V$  is a surface generated by an trivial abstract tree  $B_V \cup \emptyset$ . Now Let  $v_i^0 = v_i$   $w_j^0 = w_{j,v}$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . By Lemma 2.2.8,  $v_i^0$  intersects  $w_j^0$  in at most one point and  $\partial v_i^0 \cap w_j^0 = \emptyset$ . Specially, let  $m(0) = \{m^0\} = \emptyset$ . Thus Proposition 1 holds.

We denote by  $W_j^0$  the disk  $W_{j,v}$  for  $s(w_j) = +$ ,  $W_x^0$  the disk  $W_{x,v}$ ,  $V_i^0$  the disk  $V_i$  for  $s(v_i) = +$ ,  $V_x^0$  the disk  $V_x$ . Then, by Lemma 2.2.9, we obtain two sets of pairwise disjoint properly embedded disks  $\{V_i^0 \mid m \geq i \leq 1, s(v_i) = +\} \cup \{V_x^0\}$  in  $\mathcal{V}_+$  and  $\{W_j^0 \mid j \in \{1, \dots, n\} - m(0), s(w_j) = +\} \cup \{W_x^0\}$  in  $\mathcal{W}_+$ . By Definition 2.3.1,  $I(v_i, 0) = I(v_i)$ ,  $I(v, 0) = I(v)$ ,  $I(w_j, 0) = I(w_j) - \emptyset = I(w_j)$ ,  $I(w, 0) = I(w) - \emptyset = I(w)$ . By Lemma 2.2.9, Proposition 3 holds. Q.E.D.

**Lemma 2.5.2.** Propositions 4-6 are true for  $k = 0$ .

**Proof.** By the definitions of  $\mathcal{W}_-$  and  $\mathcal{V}_-$ ,  $B_V$  is a disk in  $\partial_+ \mathcal{V}_-$ . Hence  $B_V$  is a surface generated by an trivial abstract tree  $B_V \cup \emptyset$  in  $\partial_+ \mathcal{V}_-$ . Now Let  $c_i^0 = v_i$   $d_j^0 = w_{j,v}$ . By Lemma 2.2.8,  $c_i^0$  intersects  $d_j^0$  in at most one point and  $\partial c_i^0 \cap d_j^0 = \emptyset$ . It is easy to see that  $L(c_i^0) = L(v_i^0)$  and  $L(d_j^0) = L(w_j^0)$ . Hence Propositions 4 and 5 holds.

We denote by  $W_j^0$  the disk  $W_{j,v}$  for  $s(w_j) = -$ ,  $V_i^0$  the disk  $V_i$  for  $s(v_i) = -$ . Then, by Lemma 2.2.9, we obtain two sets of pairwise disjoint properly embedded disks  $\{V_i^0 \mid m \geq i \leq 1, s(v_i) = -\}$  in  $\mathcal{V}_-$  and  $\{W_j^0 \mid j \in \{1, \dots, n\} - m(0), s(w_j) = -\}$  in  $\mathcal{W}_-$ . Note that  $m(0) = \emptyset$ . By Definition 2.3.1,  $I(v_i, 0) = I(v_i)$ ,  $I(w_j, 0) = I(w_j)$ . By Lemma 2.2.9, Proposition 6 holds. Q.E.D.

### **The ideas of the proofs of Propositions 1-6.**

By Lemmas 2.5.1 and 2.5.2, Propositions 1-6 are true for  $k = 0$ . Now we may assume that Propositions 1-6 are true for  $k \leq l$ . We only need to prove Propositions 1-6 are true for  $k = l + 1$ . The inductive proofs depend on  $s(v_{l+1})$ .

Case 1.  $s(v_{l+1}) = +$ .

By Proposition 1,  $v_{l+1}^l$  is a properly embedded arc in  $P^l$ . Let  $m^{l+1} = \text{Min}L(v_{l+1}^l)$ . By Definition 2.3.2 and Proposition 1, either  $m^{l+1} = \emptyset$  or  $m^{l+1} \notin m(l)$ . Now let  $m(l+1) = m(l) \cup \{m^{l+1}\}$ . By Proposition 3, there is a properly embedded disk  $V_{l+1}^l$  in  $\mathcal{V}_+$  such that  $V_{l+1}^l \cap P^l = c_{l+1}^l$ . By Proposition 1,  $w_{m^{l+1}}^l$  is an arc in  $P^l$  if  $m^{l+1} \neq \emptyset$ . We can prove that  $c_{l+1}^l$  intersects  $w_{m^{l+1}}^l$  in only one point, and  $s(w_{m^{l+1}}^l) = -$ . Let  $N(\partial V_{l+1}^l)$  be a regular neighborhood of  $\partial V_{l+1}^l$  in  $\partial \mathcal{V}_+$ , and  $N(w_{m^{l+1}}^l)$  be a regular neighborhood of  $w_{m^{l+1}}^l$  in  $P^l$ . In fact,  $P^{l+1} = (P^l - N(w_{m^{l+1}}^l)) \cup N(\partial V_{l+1}^l)$ ,  $w_j^{l+1} = w_j^l$  for  $j \notin m(l+1)$ , and  $v_i^{l+1}$  ( $i \geq l+2$ ) is the band sum of  $v_i^l$  with some copies of  $\partial V_{l+1}^l$  along  $w_{m^{l+1}}^l$ . Specially,  $W_x^{l+1} = W_x^l$  and  $W_j^{l+1} = W_j^l$  for  $j \notin m(l+1)$  with  $s(w_j) = +$ ;  $V_x^{l+1}$  is the connected sum of  $V_x^l$  with some copies of  $V_{l+1}^l$ , and  $V_i^{l+1}$  is the connected sum of  $V_i^l$  with some copies of  $V_{l+1}^l$  for  $i \geq l+2$  with  $s(v_i) = +$ .

Note that Propositions 1-6 are true for  $k \leq l$ . By Propositions 2, 4 and Lemmas 2.5.1, 2.5.2, we can prove that  $L(v_i^l) = L(c_i^l)$  and  $L(d_j^l) = L(w_j^l)$ . Since  $s(w_{m^{l+1}}^l) = -$ , by Proposition 6,  $W_{m^{l+1}}^l$  is a properly embedded disk in  $\mathcal{W}_-$  such that  $W_{m^{l+1}}^l \cap F^l = d_{m^{l+1}}^l \cup_{r \in I(w_{m^{l+1}}^l, l)} d_r^l$ . We can also prove that  $d_{m^{l+1}}^l$  intersects  $c_{l+1}^l$  in only one point if  $m^{l+1} \neq \emptyset$ . Let  $N(\partial W_{m^{l+1}}^l)$  be a regular neighborhood of  $\partial W_{m^{l+1}}^l$  in  $\partial \mathcal{V}_-$ , and  $N(c_{l+1}^l)$  be a regular neighborhood of  $c_{l+1}^l$  in  $F^l$ . In fact,  $F^{l+1} = (F^l - N(c_{l+1}^l)) \cup N(\partial W_{m^{l+1}}^l)$ ,  $c_i^{l+1} = c_i^l$  for  $i \geq l+2$ , and  $d_j^{l+1}$  is the connected sum of  $d_j^l$  with some copies of  $\partial W_{m^{l+1}}^l$  along  $c_{l+1}^l$  for  $j \notin m(l+1)$ . Specially,  $V_i^{l+1} = V_i^l$  for  $i \geq l+2$  with  $s(v_i) = -$ , and  $W_j^{l+1}$  is the connected sum of  $W_j^l$  with some copies of  $W_{m^{l+1}}^l$  for  $j \notin m(l+1)$  with  $s(w_j) = -$ .

Case 2.  $s(v_{l+1}) = -$ .

Now let  $m^{l+1} = \text{Min}L(c_{l+1}^l)$ . By Proposition 4, we can prove that  $s(w_{m^{l+1}}^l) = +$ . Thus we have an alternating proof with the one of Case 1. So Propositions 4-6 are necessary to show  $s(w_{m^{l+1}}^l) = +$  when  $s(v_{l+1}) = -$ . Q.E.D.



### 3 Some properties of $\{c_i^l, d_j^l, e_\gamma^l \times I, F^l\}$

By the argument in Section 2.5, Propositions 1-6 holds for  $k = 0$ . In this chapter, we shall give some properties of  $\{v_i^l, w_j^l, b_\gamma^l \times I, P^l\}$  and  $\{c_i^l, d_j^l, e_\gamma^l \times I, F^l\}$  under the assumptions that Propositions 1-6 hold for  $k \leq l$ .

We first consider  $\{c_i^l, d_j^l, e_\gamma^l \times I, F^l\}$ .

#### 3.1 The intersection of $c_i^l$ and $d_j^l$

**Lemma 3.1.1.** Suppose that  $j \in \{1, \dots, n\} - m(l)$ , and  $j \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$  with  $s(w_\gamma) = -$ . Then  $d_j^l$  is regular in  $F^l$ .

**Proof.** Now by Proposition 4(3), if  $j < \gamma$ , then  $d_j^l$  is disjoint from  $e_\gamma^l \times I$ . By Proposition 4(2), the lemma holds. Q.E.D.

**Lemma 3.1.2.** For each  $j \in \{1, \dots, n\} - m(k)$ ,  $d_j^l = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}^l$  satisfying the following conditions:

- (1)  $j > \gamma_{i,j} \in m(l)$ , and  $e_{\gamma_{i,j}}^l$  is a core of  $e_{\gamma_{i,j}}^l \times (0, 1)$  for  $1 \leq i \leq \theta(j) - 1$ .
- (2)  $0 \leq f_{i,j} \leq l$ ,  $d_{j,f_{i,j}}^l$  is a properly embedded arc in  $E_{f_{i,j}}^l$  which is disjoint from  $\cup_{\gamma < j} \text{inte}_\gamma^l \times I$  for  $1 \leq i \leq \theta(j)$ .
- (3) For  $i \neq r$ ,  $\gamma_{i,j} \neq \gamma_{r,j}$ ,  $f_{i,j} \neq f_{r,j}$ .
- (4)  $\partial_1 e_{\gamma_{i,j}}^l = \partial_2 d_{j,f_{i,j}}^l$ ,  $\partial_2 e_{\gamma_{i,j}}^l = \partial_1 d_{j,f_{i+1,j}}^l$ ,  $\partial_1 d_j^l = \partial_1 d_{j,f_{1,j}}^l$ ,  $\partial_2 d_j^l = \partial_2 d_{j,f_{\theta(j),j}}^l$ .
- (5) For each  $f$ ,  $d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I$  intersects  $E_f^l$  in at most one component, denoted by  $d_{j,f}^l$ . Furthermore, if  $f = f_{i,j}$ , then  $d_{j,f}^l = d_{j,f_{i,j}}^l$ ; if  $f \neq f_{i,j}$  for each  $1 \leq i \leq \theta(j)$ , then  $d_{j,f}^l = \emptyset$ .

**Proof.** By Proposition 4(2),  $d_j^l$  is regular in  $\cup_{\gamma < j} e_\gamma^l \times I \cup_f E_f^l$ . Hence the lemma follows from Lemma 2.1.6. Q.E.D.

**Remark.** It is possible that  $\theta(j) = 1$ . In this case,  $d_j^l = d_{j,f_{1,j}}^l$ . Hence  $d_j^l$  is disjoint from  $e_\gamma^l \times I$  for  $\gamma < j$ .

Now by Proposition 4(4),  $d_{j,f_{i,j}}^l$  intersects  $c_i^l$  in at most one point.

**Definition 3.1.3.** We say  $i \in L(d_{j,f_{i,j}}^l)$  if  $d_{j,f_{i,j}}^l$  intersects  $c_i^l$  in one point.

**Lemma 3.1.4.** (1) If  $r \neq i$ , then  $L(d_{j,f_{i,j}}^l) \cap L(d_{j,f_{r,j}}^l) = \emptyset$ .

(2)  $L(d_j^l) = \cup_i L(d_{j,f_{i,j}}^l)$ .

**Proof.** By Proposition 4(4), if  $d_{j,f_{i,j}}^l$  intersects  $c_i^l$  in one point, then  $c_i^l \subset E_{f_{i,j}}^l$ . By Lemma 3.1.2(3), (1) holds.

By Lemma 3.1.2(5), each component of  $d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I$  is  $d_{j,f_{i,j}}^l$  for some  $f_{i,j}$ . Hence (2) holds. Q.E.D.

**Lemma 3.1.5.**  $L(v_i^l) = L(c_i^l)$  and  $L(d_j^l) = L(w_j^l)$ .

**Proof.** By Lemmas 2.5.1 and 2.5.2,  $L(w_j^0) = L(d_j^0)$ .  $L(c_i^0) = L(v_i^0)$ . Now by Propositions 2 and 5, the lemma holds. Q.E.D.

**Lemma 3.1.6.** (1) Each component of  $d_j^l \cap E_f^l$  is either  $d_{j,f}^l$  or a core of  $d_{r,f}^l \times (0, 1)$  for some  $\gamma \in m(l)$  and  $r \in I(w_\gamma, l)$ . Furthermore,  $r < \gamma < j$ ,  $s(w_\gamma) = -$ .

(2) If  $d_j^l \cap c_i^l \neq \emptyset$ , then either  $j \in L(c_i^l)$  or  $r \in L(c_i^l)$  for some  $r < j$ .

**Proof.** (1) By Lemma 3.1.2,  $d_j^l = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}^l$  such that  $\gamma_{i,j} < j$  and  $\gamma_{i,j} \in m(l)$ . Now if one component  $c$  of  $d_j^l \cap E_f^l$  is not  $d_{j,f}^l$ , then, by Lemma 3.1.2(2) and (4),  $c \subset \text{inte}_{\gamma_{i,j}}^l \cap E_f^l$  for some  $i$  where  $e_{\gamma_{i,j}}^l$  is a core of  $e_{\gamma_{i,j}}^l \times (0, 1)$ . Since  $\text{inte}_{\gamma_{i,j}}^l \cap E_f^l \neq \emptyset$ . By Proposition 4(1),  $s(w_{\gamma_{i,j}}) = -$  and  $\text{inte}_{\gamma_{i,j}}^l \times I \cap (\cup_f E_f^l \cup_{\gamma < \gamma_{i,j}} e_\gamma^l \times I) = \cup_{r \in I(w_{\gamma_{i,j}}, l)} d_r^l \times I$ . Hence  $c \subset e_{\gamma_{i,j}}^l \cap d_r^l \times I$  for  $r \in I(w_{\gamma_{i,j}}, l)$ . By Proposition 4(2),  $e_{\gamma_{i,j}}^l \cap d_r^l \times I$  is a core of  $d_r^l \times (0, 1)$ . Note that  $r < \gamma_{i,j} < j$ . By induction, (1) holds.

(2) Let  $p$  be a point in  $d_j^l \cap c_i^l$ . Then, by (1), either  $p \in d_{j,f}^l \cap c_i^l$  or  $p \in c \cap c_i^l$  where  $c$  is a core of  $d_r^l \times (0, 1)$  for some  $\gamma \in m(l)$  and  $r \in I(w_\gamma, l)$ . By Proposition 4(5),  $c_i^l$  intersects  $c$  in one point if and only if it intersects  $d_r^l = d_r^l \times \{0\}$  in one point. By induction, (2) holds. Q.E.D.

## 3.2 The intersection of $d_j^l$ and $e_\gamma^l \times I$

For each  $j \notin m(l)$  and  $\gamma \in m(l)$ , either  $j \in I(w_\gamma, l)$  or  $j \notin I(w_\gamma, l)$ . We may assume that  $j \in I(w_\gamma, l)$ . In this case, if  $j \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ , then  $\gamma_j = \emptyset$ .

**Definition 3.2.1.** (1) Let  $(d_j^l \times I)_{\gamma_j} = d_j^l \times I \subset e_{\gamma_j}^l \times I$  if  $j \in I(w_{\gamma_j}, l)$  and  $s(w_{\gamma_j}) = -$ ,  
(2) let  $(d_j^l \times I)_{\gamma_j} = d_j^l$  if  $j \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$  or  $j \in I(w_{\gamma_j}, l)$  with  $s(w_{\gamma_j}) = +$ .

Definition 3.2.1 means that  $(d_j^l \times I)_{\gamma_j}$  is either a disk or an arc.

Recalling the equality:  $d_j^l = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}^l$  for each  $j \notin m(l)$  in Lemma 3.1.2, and  $d_{j,f}^l$  in Lemma 3.1.2(5).

**Lemma 3.2.2.** If  $r \neq j$ , then  $(d_{j,f}^l \times I)_{\gamma_j}$  is disjoint from  $(d_{r,f}^l \times I)_{\gamma_r}$  for each  $0 \leq f \leq l$ .

**Proof.** Without loss of generality, we may assume that  $j \in I(w_{\gamma_j}, l)$ ,  $r \in I(w_{\gamma_r}, l)$  and  $s(w_{\gamma_j}) = s(w_{\gamma_r}) = -$ .

Assume that  $\gamma_j = \gamma_r$ . Since  $j \neq r$ ,  $(d_{j,f}^l \times I)_{\gamma_j}$  is disjoint from  $(d_{r,f}^l \times I)_{\gamma_r}$ .

Assume now that  $\gamma_j > \gamma_r$ . If  $(d_{j,f}^l \times I)_{\gamma_j} \cap (d_{r,f}^l \times I)_{\gamma_r} \neq \emptyset$ , then  $e_{\gamma_j}^l \times I \cap e_{\gamma_r}^l \times I \neq \emptyset$ . By Proposition 4(1),  $F^l$  is generated by an abstract tree. By Definition 2.1.4, each component of  $e_{\gamma_j}^l \times I \cap e_{\gamma_r}^l \times I$  is  $(c \times I)_{\gamma_j} \subset e_{\gamma_r}^l \times (0, 1)$  where  $c \subset \text{inte}_{\gamma_j}^l$  is a core of  $e_{\gamma_r}^l \times (0, 1)$ . Since  $d_{j,f}^l$  and  $d_{r,f}^l$  are properly embedded in  $E_f^l$ ,  $(d_{j,f}^l \times I)_{\gamma_j} \subset (d_{r,f}^l \times (0, 1))_{\gamma_r}$ . Hence  $d_{j,f}^l = d_{j,f}^l \times \{0\} \subset e_{\gamma_r}^l \times (0, 1)$ . By Lemma 2.2.4,  $j \notin I(w_{\gamma_r}, l)$ . By Proposition 4(3),  $j > \gamma_r$ . Note that  $d_{j,f}^l = d_{j,f}^l \times \{0\} \subset (d_{r,f}^l \times (0, 1))_{\gamma_r} \subset e_{\gamma_r}^l \times (0, 1)$ , contradicting Lemma 3.1.2(2). Q.E.D.

**Lemma 3.2.3.** If  $d_j^l \cap e_{\gamma}^l \times I \neq \emptyset$ , then either  $\gamma \leq \text{Max}\{\gamma_{1,j}, \dots, \gamma_{\theta(j)-1,j}\}$  or  $j \in I(w_{\gamma}, l)$  with  $s(w_{\gamma}) = -$ .

**Proof.** Suppose that  $\gamma \neq \gamma_{i,j}$  for  $1 \leq i \leq \theta(j) - 1$  and  $j \notin I(w_{\gamma}, l)$ . If  $\gamma > j$ , then, by Proposition 4(3),  $d_j^l$  is disjoint from  $e_{\gamma}^l \times I$ .

Suppose that  $\gamma < j$ . Then, by Lemma 3.1.2,  $d_{j,f_{i,j}}^l$  is disjoint from  $\text{inte}_{\gamma}^l \times I$  for each  $1 \leq i \leq \theta(j)$ . By Proposition 4(2), each component of  $d_j^l \cap e_{\gamma}^l \times I$  is a core of  $e_{\gamma}^l \times (0, 1)$ . Since  $F^l$  is generated by an abstract tree,  $\partial e_{\gamma}^l \times I \cap \partial e_{\lambda}^l \times I = \emptyset$  for each  $\gamma \neq \lambda \in m(l)$ . Since  $\gamma \neq \gamma_{i,j}$ , each component of  $d_j^l \cap e_{\gamma}^l \times I$  lies in  $\text{inte}_{\gamma_{i,j}}^l$ . (See the equality of  $d_j^l$  in Lemma 3.1.2.) Thus each component of  $e_{\gamma_{i,j}}^l \cap e_{\gamma}^l \times I$  is a core of  $e_{\gamma}^l \times I$ . By Definition 2.1.4,  $\gamma < \gamma_{i,j}$ . Q.E.D.

**Lemma 3.2.4.** If  $\text{inte}_{\gamma}^l \times I \cap E_f^l \neq \emptyset$  for some  $\gamma \in m(l)$  and some  $0 \leq f \leq l$ , then each component of  $\text{inte}_{\gamma}^l \times I \cap E_f^l$  is either  $(d_{r,f}^l \times I)_{\gamma}$  for some  $r \in I(w_{\gamma}, l)$  or  $(c \times I)_{\gamma} \subset (d_{j,f}^l \times (0, 1))_{\gamma_j}$  where  $c \subset \text{inte}_{\gamma}^l$  is a core of  $d_{j,f}^l \times (0, 1)$  for some  $\gamma_j \in m(l)$  and  $j \in I(w_{\gamma_j}, l)$ . Furthermore,  $j < \gamma_j < \gamma$ ,  $s(w_{\gamma}) = s(w_{\gamma_j}) = -$ .

**Proof.** Now by Proposition 4(1),  $s(w_\gamma) = -$  and if  $C$  is a component of  $\text{inte}_\gamma^l \times I \cap E_f^l$ , then  $C \subset (d_r^l \times I)_\gamma$  for some  $r \in I(w_\gamma, l)$ . Hence  $r < \gamma$ . If  $C \neq (d_{r,f}^l \times I)_\gamma$ . Then, by Lemma 3.1.2,  $C \subset \text{inte}_{\gamma_i,r}^l \times I \cap E_f^l$  for some  $1 \leq i \leq \theta(r) - 1$ . By induction, the lemma holds. Q.E.D.

**Corollary 3.2.5.** If  $L(c_{l+1}^l) = \emptyset$ , then  $d_j^l$  and  $e_\gamma^l \times I$  are disjoint from  $c_{l+1}^l$  for each  $j \in \{1, \dots, n\} - m(l)$  and  $\gamma \in m(l)$ .

The corollary follows from Lemma 3.1.6 and Lemma 3.2.4 and Proposition 4(5).

**Lemma 3.2.6.** (1) Suppose that  $j < \gamma < \gamma_j$  and  $j \in I(w_{\gamma_j}, l)$ . Then  $(d_j^l \times I)_{\gamma_j}$  is disjoint from  $e_\gamma^l \times I$ .

(2) Suppose that  $j > \gamma$ . Then  $(\text{int}d_{j,f}^l \times I)_{\gamma_j}$  is disjoint from  $e_\gamma^l \times I$ .

**Proof.** (1) Suppose that  $s(w_{\gamma_j}) = +$ . Then, by Proposition 4(3) and Definition 3.2.1,  $(d_j^l \times I)_{\gamma_j} = d_j^l$  is disjoint from  $e_\gamma^l \times I$ .

Now suppose that  $s(w_{\gamma_j}) = -$ . By Proposition 4(1) and Definition 2.1.4,  $e_{\gamma_j}^l \times I \cap e_\gamma^l \times I$  is  $(c \times I)_{\gamma_j}$  where  $c \subset \text{inte}_{\gamma_j}^l$  is a core of  $e_\gamma^l \times (0, 1)$ . By Lemma 3.2.3,  $(d_j^l \times I)_{\gamma_j}$  is disjoint from  $e_\gamma^l \times I$ .

(2) Suppose that  $s(w_{\gamma_j}) = +$ . Then, by Lemma 3.1.2 and Definition 3.2.1,  $(\text{int}d_j^l \times I)_{\gamma_j} = \text{int}d_j^l$  is disjoint from  $e_\gamma^l \times I$ .

Now suppose that  $s(w_{\gamma_j}) = -$ . By Proposition 4(1) and Definition 2.1.4, each component of  $e_{\gamma_j}^l \times I \cap e_\gamma^l \times I$  is  $(c \times I)_{\gamma_j} \subset e_\gamma^l \times (0, 1)$ , where  $c \subset \text{inte}_{\gamma_j}^l$  is a core of  $e_\gamma^l \times (0, 1)$ . Since  $\text{int}d_j^l$  is disjoint from  $e_\gamma^l \times I$ , (2) holds. Q.E.D.

### 3.3 Properties of $d_j^l$ for $j \in L(c_{l+1}^l)$

**Definition 3.3.0.** Let  $m^{l+1} = \text{Min}L(c_{l+1}^l)$ .

By Proposition 4(4),  $c_{l+1}^l$  lies in  $E_f^l$  for some  $0 \leq f \leq l$ . Without loss of generality, we may assume that  $c_{l+1}^l \subset E_0^l$ . Recalling  $d_{j,f}^l = (d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I) \cap E_f^l$  defined in Lemma 3.1.2(5). Now if  $j \in L(c_{l+1}^l)$ , then, by Proposition 4(4) and Lemma 3.1.2,  $d_{j,0}^l$  intersects  $c_{l+1}^l$  in one point and  $j \geq m^{l+1}$ .

Now we rearrange all the elements in  $L(c_{l+1}^l)$  as  $\dots, j_{-1}, j_0 = m^{l+1}, j_1, \dots$  according to

the order of  $(\cup_{j \in L(c_{l+1}^l)} d_{j,0}^l) \cap c_{l+1}^l$  lying in  $c_{l+1}^l$  as in Figure 6.

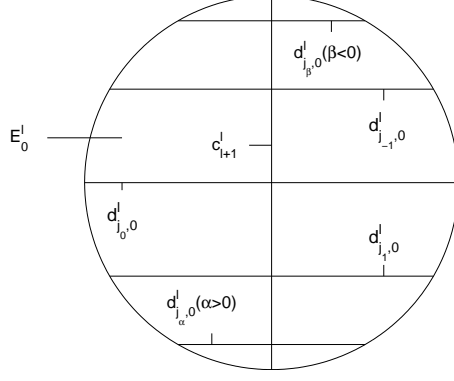


Figure 6

**Lemma 3.3.1.** For each  $j_\alpha \in L(c_{l+1}^l)$ ,  $d_{j_\alpha}^l = d_{0,\alpha} \cup_{i=\delta(\alpha)}^{\theta(\alpha)} d_{i,\alpha} \cup_{i=\delta(\alpha)}^{\theta(\alpha)} e_{i,\alpha}$  satisfying the following conditions:

- (1) For  $\delta(\alpha) \leq i \neq 0 \leq \theta(\alpha)$ ,  $e_{i,\alpha}$  is a core of  $e_{\gamma_{i,\alpha}}^l \times (0, 1)$  for some  $\gamma_{i,\alpha} < j_\alpha$ .
- (2) For  $\delta(\alpha) \leq i \neq 0 \leq \theta(\alpha)$  and  $d_{i,\alpha}$  is a properly embedded arc in  $E_{f_{i,\alpha}}^l$  which is disjoint from  $\cup_{\gamma < j_\alpha} \text{inte}_\gamma^l \times I$ .
- (3)  $d_{0,\alpha}$  is a properly embedded arc in  $E_0^l$ .
- (4) For  $i \neq r \neq 0$ ,  $\gamma_{i,\alpha} \neq \gamma_{r,\alpha}$ ,  $f_{i,\alpha} \neq f_{r,\alpha}$ , and  $f_{i,\alpha} \neq 0$ .
- (5) For  $\delta(\alpha) \leq i \neq 0 \leq \theta(\alpha)$ ,  $\partial_1 e_{i,\alpha} = \partial_2 d_{i,\alpha}$ ,  $\partial_2 e_{i,\alpha} = \partial_1 d_{i+1,\alpha}$ ,  $\partial_1 e_{1,\alpha} = \partial_2 d_{0,\alpha}$ ,  $\partial_2 e_{-1,\alpha} = \partial_1 d_{0,\alpha}$ ,  $\partial_1 d_{j_\alpha}^l = \partial_1 d_{\delta(\alpha),\alpha}$ ,  $\partial_2 d_{j_\alpha}^l = \partial_2 d_{\theta(\alpha),\alpha}$ .

**Proof.** Since  $j_\alpha \in L(c_{l+1}^l)$ ,  $d_{j_\alpha,0}^l \neq \emptyset$ , by Lemma 3.1.2,  $d_{j_\alpha}^l = \cup_{i=\delta(j_\alpha)}^{-1} d_{j_\alpha, f_{i,j_\alpha}}^l \cup d_{j_\alpha,0}^l \cup_{i=1}^{\theta(j_\alpha)} d_{j_\alpha, f_{i,j_\alpha}}^l \cup_{i=\delta(j_\alpha)}^{-1} e_{\gamma_{i,j_\alpha}} \cup_{i=1}^{\theta(j_\alpha)} e_{\gamma_{i,j_\alpha}}$ . Now we denote by  $\delta(\alpha)$  the integer  $\delta(j_\alpha)$ ,  $\theta(\alpha)$  the integer  $\theta(j_\alpha)$ ,  $d_{i,\alpha}$  the arc  $d_{j_\alpha, f_{i,j_\alpha}}^l$ ,  $e_{i,\alpha}$  the arc  $e_{\gamma_{i,j_\alpha}}$  for  $i \neq 0$ . Let  $f_{i,\alpha} = f_{i,j_\alpha}$  and  $\gamma_{i,\alpha} = \gamma_{i,j_\alpha}$ . In particular, we denote by  $d_{0,\alpha}$  the arc  $d_{j_\alpha,0}^l$ . By Lemma 3.1.2, the lemma holds. Q.E.D.

**Remark 3.3.2.** 1) In order to simplify the formulation, we shall write as  $d_{j_\alpha}^l = \cup_{i=\delta(\alpha)}^{\theta(\alpha)} d_{i,\alpha} \cup_{i=\delta(\alpha)}^{\theta(\alpha)} e_{i,\alpha}$ . In the formulation, " $\cup_{i=\delta(\alpha)}^{\theta(\alpha)} d_{i,\alpha}$ " means "the union index from  $\delta(\alpha)$  to  $-1$ , then  $0$ , then  $1$ , and then to  $\theta(\alpha)$ ", but " $\cup_{i=\delta(\alpha)}^{\theta(\alpha)} e_{i,\alpha}$ " means "the union index from  $\delta(\alpha)$  to  $-1$ , then  $1$ , and then to  $\theta(\alpha)$ ".

- 2) If  $\delta(\alpha) = \theta(\alpha) = 0$ , then  $d_{j_\alpha}^l = d_{0,\alpha}$ .

**Lemma 3.3.3.** (1)  $d_j^l \cap c_{l+1}^l \subset (\cup_\alpha (d_{0,\alpha} \times I)_{\gamma_\alpha}) \cap c_{l+1}^l$  where  $j \in \{1, \dots, n\} - m(l)$ ,  $j_\alpha \in L(c_{l+1}^l)$ , and  $(d_{0,\alpha} \times I)_{\gamma_\alpha}$  is defined in Definition 3.2.1.

(2)  $e_\gamma^l \times I \cap c_{l+1}^l \subset (\cup_\alpha (d_{0,\alpha} \times I)_{\gamma_\alpha}) \cap c_{l+1}^l$  for  $\gamma \in m(l)$ .

**Proof.** (1) Now each component of  $d_j^l \cap c_{l+1}^l$  is a point. By assumption,  $c_{l+1}^l \subset E_0^l$ . If  $d_j^l \cap c_{l+1}^l \neq \emptyset$ , then  $d_j^l \cap c_{l+1}^l = (d_j^l \cap E_0^l) \cap c_{l+1}^l$ . By Lemma 3.1.6, each component of  $d_j^l \cap E_0^l$  is either  $d_{j,0}^l$  or a core of  $d_{r,0}^l \times (0, 1)$  for some  $\gamma \in m(l)$  and  $r \in I(w_\gamma, l)$ . Suppose that  $d$  is a component of  $d_j^l \cap E_0^l$  such that  $d \cap c_{l+1}^l \neq \emptyset$ . If  $d = d_{j,0}^l$ , then  $j \in L(c_{l+1}^l)$ . If  $d = d_{r,0}^l$ , then  $r \in L(c_{l+1}^l)$ .

(2) follows from Lemma 3.2.4 and Proposition 4(5). Q.E.D.

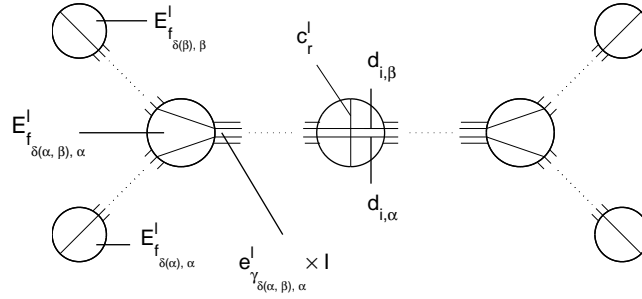


Figure 7

**Lemma 3.3.4.** If  $j_\alpha, j_\beta \in L(c_{l+1}^l)$ , then there are two integers  $\delta(\alpha, \beta) \leq 0$  and  $\theta(\alpha, \beta) \geq 0$  satisfying the following conditions:

- (1)  $\delta(\alpha), \delta(\beta) \leq \delta(\alpha, \beta) \leq \theta(\alpha, \beta) \leq \theta(\alpha), \theta(\beta)$ .
- (2) If  $\delta(\alpha, \beta) \leq i \leq \theta(\alpha, \beta)$ , then  $\gamma_{i,\alpha} = \gamma_{i,\beta}$  and  $f_{i,\alpha} = f_{i,\beta}$ .
- (3)  $\{\gamma_{i,\alpha} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} \cap \{\gamma_{i,\beta} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} = \emptyset$ .
- (4)  $\{f_{i,\alpha} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} \cap \{f_{i,\beta} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} = \emptyset$ .
- (5) If  $\delta(\alpha, \beta) + 1 \leq i \leq \theta(\alpha, \beta) - 1$ , then  $L(d_{i,\alpha}) = L(d_{i,\beta})$ .

**Proof.** Since  $j_\alpha, j_\beta \in L(c_{l+1}^l)$ ,  $d_{0,\alpha}, d_{0,\beta} \neq \emptyset$ . Suppose that  $\delta(\alpha, \beta)$  and  $\theta(\alpha, \beta)$  are two integers such that  $f_{i,\alpha} = f_{i,\beta}$  for  $\delta(\alpha, \beta) \leq i \leq \theta(\alpha, \beta)$ . Since  $\cup_f E_f^l \cup e_\gamma^l$  is an abstract tree,  $\gamma_{i,\alpha} = \gamma_{i,\beta}$  for  $\delta(\alpha, \beta) \leq i \leq \theta(\alpha, \beta)$ . Now  $\delta(\alpha), \delta(\beta) \leq \delta(\alpha, \beta) \leq \theta(\alpha, \beta) \leq \theta(\alpha), \theta(\beta)$ .

Assume now that  $\gamma_{\delta(\alpha,\beta)-1,\alpha} \neq \gamma_{\delta(\alpha,\beta)-1,\beta}$  and  $\gamma_{\theta(\alpha,\beta)+1,\alpha} \neq \gamma_{\theta(\alpha,\beta)+1,\beta}$ , and  $\gamma_{i,\alpha} = \gamma_{r,\beta}$  for some  $i < \delta(\alpha, \beta) - 1$  or  $i > \theta(\alpha, \beta) + 1$  and some  $r < \delta(\alpha, \beta) - 1$  or  $r > \theta(\alpha, \beta) + 1$ .

Since  $d_{0,\alpha}, d_{0,\beta} \neq \emptyset$ ,  $\cup_{\gamma \in m(l)} e_\gamma \cup_{f=0}^l E_f^l$  is not an abstract tree even if  $\delta(\alpha, \beta) = \theta(\alpha, \beta) = 0$ , contradicting Proposition 4(1). Hence (1), (2), (3) and (4) hold.

Suppose that  $\delta(\alpha, \beta) + 1 \leq i \leq -1$ . Now  $f_{i,\alpha} = f_{i,\beta}$  and  $\gamma_{i-1,\alpha} = \gamma_{i-1,\beta}$ ,  $\gamma_{i,\alpha} = \gamma_{i,\beta}$ . Hence  $\partial d_{i,\alpha} \cup \partial d_{i,\beta} \subset (\partial e_{\gamma_{i-1,\alpha}}^l \cup \partial e_{\gamma_{i,\alpha}}^l) \times I$ . By Proposition 4(4) and (5),  $\partial c_r^l$  is disjoint from  $(e_{\gamma_{i-1,\alpha}}^l \cup e_{\gamma_{i,\alpha}}^l) \times I$ . Hence  $c_r^l$  intersects  $d_{i,\alpha}$  in one point if and only if  $c_r^l$  intersects  $d_{i,\beta}$  in one point as in Figure 7. Hence (5) holds. Q.E.D.

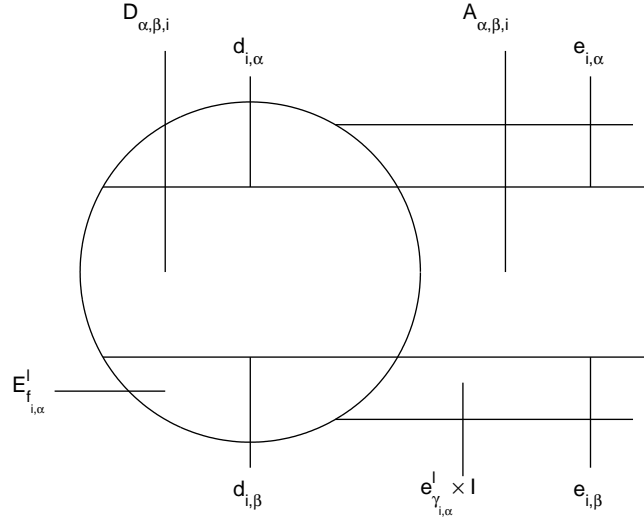


Figure 8(a)

### 3.4 $\delta(\alpha, \beta)$ and $\theta(\alpha, \beta)$

Recalling the equality  $d_{j_\alpha}^l = \cup_{i=\delta(\alpha)}^{\theta(\alpha)} d_{i,\alpha} \cup_{i=\delta(\alpha)}^{\theta(\alpha)} e_{i,\alpha}$  in Remark 3.3.2, and the two integers  $\delta(\alpha, \beta)$  and  $\theta(\alpha, \beta)$  in Lemma 3.3.4. In this case,  $d_{i,\alpha}$  is an arc in  $E_{f_{i,\alpha}}^l$  and  $e_{i,\alpha}$  is a core of  $e_{\gamma_{i,\alpha}}^l \times (0, 1)$  for  $i \neq 0$ . In particular,  $d_{0,\alpha}$  is an arc in  $E_0^l$ . By Lemma 3.3.4,  $f_{i,\alpha} = f_{i,\beta}$  and  $\gamma_{i,\alpha} = \gamma_{i,\beta}$  for  $\delta(\alpha, \beta) \leq i \neq 0 \leq \theta(\alpha, \beta)$ .

**Definition 3.4.1.** For each  $\delta(\alpha, \beta) \leq i \neq 0 \leq \theta(\alpha, \beta)$ .

(1) Let  $D_{\alpha,\beta,i}$  be the disk in  $E_{f_{i,\alpha}}^l$  which is bounded by  $d_{i,\alpha}$  and  $d_{i,\beta}$  with two arcs in  $\partial E_{f_{i,\alpha}}^l$ .

(2) Let  $A_{\alpha,\beta,i}$  be the disk in  $e_{\gamma_{i,\alpha}}^l \times I$  which is bounded by  $e_{i,\alpha}$  and  $e_{i,\beta}$  with two arcs in  $(\partial e_{\gamma_{i,\alpha}}^l) \times I$  as in Figure 8(a).

(3) Let  $D_{\alpha,\beta,0}$  be the disk in  $E_0^l$  which is bounded by  $d_{0,\alpha}$  and  $d_{0,\beta}$  with two arcs in  $\partial E_0^l$  as in Figure 8(a).

(4) Let  $D_{\alpha,\beta} = D_{\alpha,\beta,0} \cup_{i=\delta(\alpha,\beta)+1}^{\theta(\alpha,\beta)-1} D_{\alpha,\beta,i} \cup_{i=\delta(\alpha,\beta)}^{\theta(\alpha,\beta)} A_{\alpha,\beta,i}$ . Now we rewrite  $D_{\alpha,\beta}$  as  $\cup_{i=\delta(\alpha,\beta)+1}^{\theta(\alpha,\beta)-1} D_{\alpha,\beta,i} \cup_{i=\delta(\alpha,\beta)}^{\theta(\alpha,\beta)} A_{\alpha,\beta,i}$ .

Now suppose that  $\alpha < \lambda < \beta$ . In this section, we shall prove that  $\delta(\alpha, \lambda) \leq \delta(\alpha, \beta) \leq \theta(\alpha, \beta) \leq \theta(\alpha, \lambda)$ , and  $D_{\alpha,\beta}$  is a disk in  $F^l$ .

**Lemma 3.4.2.** Let  $j \in \{1, \dots, n\} - m(l)$ . If  $d_j^l \cap D_{\alpha,\alpha+1,i} \neq \emptyset$  for some  $\delta(\alpha, \alpha+1) < i < \theta(\alpha, \alpha+1)$ , then  $d_j^l \cap D_{\alpha,\alpha+1,i} \subset (d_{i,\alpha} \times I)_{\gamma_\alpha} \cup (d_{i,\alpha+1} \times I)_{\gamma_{\alpha+1}}$ . Furthermore, if  $j \neq j_\alpha, j_{\alpha+1}$ , then  $j > \gamma_\alpha$  or  $\gamma_{\alpha+1}$ .

**Proof.** Assume that  $d_j^l \cap D_{\alpha,\alpha+1,i} \neq \emptyset$  for some  $\delta(\alpha, \alpha+1) < i \leq 0$ . By Lemma 3.1.6, each component of  $d_j^l \cap D_{\alpha,\alpha+1,i}$  is either  $d_{j,f_{i,\alpha}}^l$  or a copy of  $d_{r,f_{i,\alpha}}^l$  for some  $r \in I(w_\gamma, l)$  and  $\gamma \in m(l)$  with  $\gamma < j$ . There are two cases:

Case 1. one component  $c$  of  $d_j^l \cap D_{\alpha,\alpha+1,i}$  is  $d_{j,f_{i,\alpha}}^l$ .

Now if  $j = j_\alpha$  or  $j_{\alpha+1}$ , then, by Lemma 3.3.2, either  $d_{j,f_{i,\alpha}}^l = d_{i,\alpha}$  or  $d_{j,f_{i,\alpha}}^l = d_{i,\alpha+1}$ .

Suppose that  $j \neq j_\alpha, j_{\alpha+1}$ . By Lemma 3.2.2,  $d_{j,f_{i,\alpha}}^l = c$  is disjoint from  $(d_{i,\alpha} \times I)_{\gamma_\alpha} \cup (d_{i,\alpha+1} \times I)_{\gamma_{\alpha+1}}$ . Since  $\delta(\alpha, \alpha+1) < i \leq 0$ ,  $\gamma_{i-1,\alpha} = \gamma_{i-1,\alpha+1}$  and  $\gamma_{i,\alpha} = \gamma_{i,\alpha+1}$ . Now  $\partial d_{j,f_{i,\alpha}}^l \subset (\partial e_{\gamma_{i-1,\alpha}}^l \cup \partial e_{\gamma_{i,\alpha}}^l) \times I$ .

Now we claim that  $\partial_1 d_{j,f_{i,\alpha}}^l \subset \partial e_{\gamma_{i-1,\alpha}}^l \times I$  and  $\partial_2 d_{j,f_{i,\alpha}}^l \subset (\partial e_{\gamma_{i,\alpha}}^l) \times I$ .

Suppose that  $\partial d_{j,f_{i,\alpha}}^l \subset (\partial e_{\gamma_{i-1,\alpha}}^l) \times I$ , then  $d_j^l \cap (\partial e_{\gamma_{i-1,\alpha}}^l) \times I \neq \emptyset$ . By Proposition 4(1),  $j \notin I(w_{\gamma_{i-1,\alpha}}, l)$ . By Proposition 4(3),  $j > \gamma_{i-1,\alpha}$ . By Proposition 4(2),  $d_j^l$  is regular in  $\cup_f E_f^l \cup_{\gamma < j} e_\gamma^l \times I$ . This means that  $d_j^l - \cup_f d_{j,f}^l$  intersects  $e_{\gamma_{i-1,\alpha}}^l \times I$  in at least two cores of  $e_{\gamma_{i-1,\alpha}}^l \times I$ , contradicting Lemma 3.1.2(3).

Now  $j > \gamma_{i-1,\alpha}, \gamma_{i,\alpha}$ ,  $\partial_1 d_{j,f_{i,\alpha}}^l \subset \partial e_{\gamma_{i-1,\alpha}}^l \times I$  and  $\partial_2 d_{j,f_{i,\alpha}}^l \subset (\partial e_{\gamma_{i,\alpha}}^l) \times I$ . Hence  $d_j^l$  intersects  $A_{\alpha,\alpha+1,i}$  in at least a core of  $A_{\alpha,\alpha+1,i}$ . By Lemma 3.1.2,  $d_{j,f_{i+1,\alpha}}^l \neq \emptyset \subset D_{\alpha,\alpha+1,i+1}$  as in Figure 8(b). By induction,  $d_{j,0}^l \neq \emptyset \subset D_{\alpha,\alpha+1,0}$ , and  $j \in L(c_{l+1}^l)$ . Contradicting the order of  $j_\alpha, j_{\alpha+1}$  we rearrange.

Case 2. one component  $c$  of  $d_j^l \cap D_{\alpha,\alpha+1,i}$  is a copy of  $d_{r,f_{i,\alpha}}^l$ .

By the argument in Case 1,  $r = j_\alpha$  or  $j_{\alpha+1}$  and  $c \subset (d_{i,\alpha} \times I)_{\gamma_\alpha} \cup (d_{i,\alpha+1} \times I)_{\gamma_{\alpha+1}}$ . Assume



now that  $c \subset (d_{i,\alpha} \times I)_{\gamma_\alpha}$ . Since  $c \subset d_j^l$  and  $j \neq j_\alpha$ . Then, by Proposition 4(3),  $j > \gamma_\alpha$ . Q.E.D.

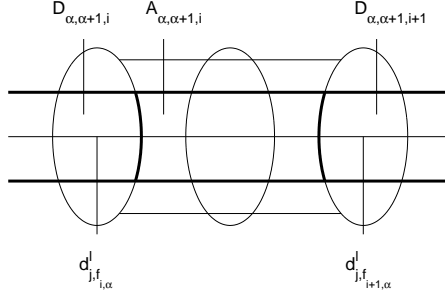


Figure 8(b)

**Lemma 3.4.3.** If  $\text{inte}_\gamma^l \times I \cap D_{\alpha,\alpha+1,i} \neq \emptyset$  for some  $\delta(\alpha, \alpha+1) < i < \theta(\alpha, \alpha+1)$ , then each component of  $\text{inte}_\gamma^l \times I \cap D_{\alpha,\alpha+1,i}$  lies in one of  $(d_{i,\alpha} \times I)_{\gamma_\alpha}$  and  $(d_{i,\alpha+1} \times I)_{\gamma_{\alpha+1}}$ . Furthermore, if one component of  $\text{inte}_\gamma^l \times I \cap D_{\alpha,\alpha+1,i}$  lies in  $(d_{i,\alpha} \times I)_{\gamma_\alpha}$ , then  $\gamma \geq \gamma_\alpha$ .

**Proof.** Suppose that  $\text{inte}_\gamma^l \times I \cap D_{\alpha,\alpha+1,i} \neq \emptyset$ . By Lemma 3.2.4, each component of  $\text{inte}_\gamma^l \times I \cap E_{f_{i,\alpha}}^l$  is contained in  $(d_{r,f_{i,\alpha}}^l \times I)_{\gamma_r}$  for some  $\gamma_r \in m(l)$  and  $r \in I(w_{\gamma_r}, l)$ . Furthermore,  $\gamma \geq \gamma_r > r$ . If  $r \neq j_\alpha, j_{\alpha+1}$ , then, by Lemma 3.2.2 and Definition 3.2.1,  $(d_{i,\alpha} \times I)_{\gamma_\alpha}$  and  $(d_{i,\alpha+1} \times I)_{\gamma_{\alpha+1}}$  are disjoint from  $(d_{r,f_{i,\alpha}}^l \times I)_{\gamma_r}$ . In this case, if  $(d_{r,f_{i,\alpha}}^l \times I)_{\gamma_r} \cap D_{\alpha,\alpha+1,i} \neq \emptyset$ , then  $d_{r,f_{i,\alpha}}^l \subset D_{\alpha,\alpha+1,i}$ . By the proof of Lemma 3.4.2, this is impossible. Q.E.D.

**Lemma 3.4.4.** If  $\delta(\alpha, \alpha+1) \leq i \neq r \neq 0 \leq \theta(\alpha, \alpha+1)$ , then  $A_{\alpha,\alpha+1,i}$  is disjoint from  $A_{\alpha,\alpha+1,r}$ .

**Proof.** By Lemma 3.3.1,  $\gamma_{i,\alpha} \neq \gamma_{r,\alpha}$ . We may assume that  $\gamma_{i,\alpha} < \gamma_{r,\alpha}$ . If  $e_{\gamma_{r,\alpha}}^l \times I$  is disjoint from  $e_{\gamma_{i,\alpha}}^l \times I$ , then  $A_{\alpha,\alpha+1,r} \subset e_{\gamma_{r,\alpha}}^l \times I$  is disjoint from  $A_{\alpha,\alpha+1,i} \subset e_{\gamma_{i,\alpha}}^l \times I$ .

Suppose now that  $e_{\gamma_{r,\alpha}}^l \times I \cap e_{\gamma_{i,\alpha}}^l \times I \neq \emptyset$ . Now by Proposition 4(1) and Definition 2.1.4, each component of  $A_{\alpha,\alpha+1,r} \cap e_{\gamma_{i,\alpha}}^l \times I$  is a disk in  $e_{\gamma_{i,\alpha}}^l \times I$ . If one component of  $A_{\alpha,\alpha+1,r} \cap e_{\gamma_{i,\alpha}}^l \times I$ , say  $A$ , is not disjoint from  $A_{\alpha,\alpha+1,i}$ , then either one of the two arcs  $A \cap e_{r,\alpha}$  and  $A \cap e_{r,\alpha+1}$  lies in  $A_{\alpha,\alpha+1,i}$  as in Figure 9(a) or one of  $e_{i,\alpha}$  and  $e_{i,\alpha+1}$  lies in  $A$  as in Figure 9(b). Here  $e_{r,\alpha}$  is defined in Definition 3.4.1.

We first suppose that one of  $A \cap e_{r,\alpha}$  and  $A \cap e_{r,\alpha+1}$ , say  $A \cap e_{r,\alpha}$ , lies in  $A_{\alpha,\alpha+1,i}$ . By Definition 2.1.4,  $(\partial e_{\gamma_{r,\alpha}}^l) \times I \cap (\partial e_{\gamma_{i,\alpha}}^l) \times I = \emptyset$ . Hence  $e_{r,\alpha} \cap \text{int} D_{\alpha,\alpha+1,i+1} \neq \emptyset$ . By Lemma

3.2.4, there is  $j$  such that a copy of  $d_{j,f_{i+1},\alpha}^l$  is contained in  $D_{\alpha,\alpha+1,i+1}$ , and  $j < \gamma_{i,\alpha} < j_\alpha, j_{\alpha+1}$ , contradicting Lemma 3.4.2.

Assume that one of  $e_{i,\alpha}$  and  $e_{i,\alpha+1}$ , say  $e_{i,\alpha}$ , lies in  $A$  as in Figure 9(b). Then  $d_{i+1,\alpha} \subset A_{\alpha,\alpha+1,r} \subset e_{\gamma_{r,\alpha}}^l \times I$ . Note that  $j_\alpha > \gamma_{r,\alpha}$ . By Lemma 3.1.2(2), this is impossible. Q.E.D.

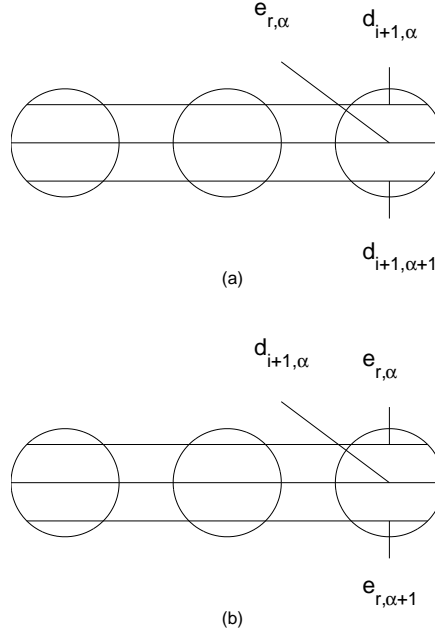


Figure 9

**Lemma 3.4.5.** For each  $\delta(\alpha, \alpha + 1) \leq i \leq \theta(\alpha, \alpha + 1)$  and  $\delta(\alpha, \alpha + 1) + 1 \leq r \neq 0 \leq \theta(\alpha, \alpha + 1) - 1$ ,  $\text{inte}_{\gamma_{i,\alpha}}^l \times I \cap A_{\alpha,\alpha+1,i}$  is disjoint from  $D_{\alpha,\alpha+1,r}$ . Furthermore, if  $i \neq r - 1, r$ , then  $A_{\alpha,\alpha+1,i}$  is disjoint from  $D_{\alpha,\alpha+1,r}$ .

**Proof.** By Lemma 3.2.4, each component of  $\text{inte}_{\gamma_{i,\alpha}}^l \times I \cap E_{f_{r,\alpha}}^l$  is contained in  $(d_{j,f_{r,\alpha}}^l \times I)_\gamma$ , where  $\gamma \leq \gamma_{i,\alpha} \in m(l)$  and  $j \in I(w_\gamma, l)$ . Hence  $j < \gamma_{i,\alpha}$ . Since  $\gamma_{i,\alpha} < j_\alpha, j_{\alpha+1}$ , by Lemma 3.3.1,  $d_{r,\alpha}$  and  $d_{r,\alpha+1}$  are disjoint from  $\text{inte}_{\gamma_{i,\alpha}}^l \times I$ . Thus if  $\text{inte}_{\gamma_{i,\alpha}}^l \times I \cap D_{\alpha,\alpha+1,r} \neq \emptyset$ , then one component of  $\text{inte}_{\gamma_{i,\alpha}}^l \times I \cap E_{f_{r,\alpha}}^l$  lies in  $D_{\alpha,\alpha+1,r}$ . Hence  $(d_{j,f_{r,\alpha}}^l \times I)_\gamma$  lies in  $D_{\alpha,\alpha+1,r}$  for some  $j < j_\alpha, j_{\alpha+1}$ . By Lemma 3.4.2, this is impossible.

Note that  $\partial_1 d_{r,\alpha}, \partial_1 d_{r,\alpha+1} \subset (\partial e_{\gamma_{r-1},\alpha}^l) \times I$  and  $\partial_2 d_{r,\alpha}, \partial_2 d_{r,\alpha+1} \subset (\partial e_{\gamma_{r,\alpha}}^l) \times I$ . If  $i \neq r, r - 1$ , then, by Lemma 2.1.4,  $(\partial e_{\gamma_{i,\alpha}}^l) \times I \cap ((\partial e_{\gamma_{r-1},\alpha}^l) \times I \cup (\partial e_{\gamma_{r,\alpha}}^l) \times I) = \emptyset$ . Q.E.D.

**Lemma 3.4.6.** (1) If  $d_j^l \cap A_{\alpha, \alpha+1, i} \neq \emptyset$ , then  $d_j^l \cap A_{\alpha, \alpha+1, i} \subset (e_{i, \alpha} \times I)_{\gamma_\alpha} \cup (e_{i, \alpha+1} \times I)_{\gamma_{\alpha+1}}$ .

Furthermore, if  $j \neq j_\alpha, j_{\alpha+1}$ , then  $j > \gamma_\alpha$  or  $\gamma_{\alpha+1}$ .

(2) If  $\gamma > \gamma_{i, \alpha}$  for  $\delta(\alpha, \alpha+1) \leq i \leq \theta(\alpha, \alpha+1)$  and  $e_\gamma^l \times I \cap A_{\alpha, \alpha+1, i} \neq \emptyset$ , then  $e_\gamma^l \times I \cap A_{\alpha, \alpha+1, i} \subset (e_{i, \alpha} \times I)_{\gamma_\alpha} \cup (e_{i, \alpha+1} \times I)_{\gamma_{\alpha+1}}$ . Furthermore,  $\gamma \geq \gamma_\alpha$  or  $\gamma_{\alpha+1}$ .

**Proof.** (1) By Lemma 3.3.1 and Definition 3.4.1,  $e_{i, \alpha}, e_{i, \alpha+1}, A_{\alpha, \alpha+1, i} \subset e_{\gamma_{i, \alpha}}^l \times (0, 1)$ . If  $d_j^l \cap A_{\alpha, \alpha+1, i} \neq \emptyset$ , then  $j \notin I(w_{\gamma_{i, \alpha}}, l)$ ; otherwise,  $d_j^l \subset e_{\gamma_{i, \alpha}}^l \times \{0\}$ . By Proposition 4(3),  $j > \gamma_{i, \alpha}$ . By Proposition 4(2) and Definition 2.1.5, each component of  $d_j^l \cap e_{\gamma_{i, \alpha}}^l \times I$  is a core of  $e_{\gamma_{i, \alpha}}^l \times I$  which lies in  $\text{int} d_j^l$ . Thus  $d_j^l \cap D_{\alpha, \alpha+1, i+1} \neq \emptyset$ . By Lemma 3.4.2, (1) holds.

(2) Now suppose that  $\gamma > \gamma_{i, \alpha}$  and  $e_\gamma^l \times I \cap A_{\alpha, \alpha+1, i} \neq \emptyset$ . Then each component of  $e_\gamma^l \times I \cap e_{\gamma_{i, \alpha}}^l \times I$  is  $(c \times I)_\gamma \subset e_{\gamma_{i, \alpha}}^l \times I$  where  $c$  is a core of  $e_{\gamma_{i, \alpha}}^l \times I$ . By Definition 2.1.4,  $(\partial e_\gamma^l) \times I \cap (\partial e_{\gamma_{i, \alpha}}^l) \times I = \emptyset$ . Hence  $\text{int} e_\gamma^l \times I \cap D_{\alpha, \alpha+1, i+1} \neq \emptyset$ . By Lemma 3.4.3,  $\text{int} e_\gamma^l \times I \cap D_{\alpha, \alpha+1, i+1} \subset (d_{i, \alpha} \times I)_{\gamma_\alpha} \cup (d_{i, \alpha+1} \times I)_{\gamma_{\alpha+1}}$ . Furthermore, If  $e_\gamma^l \times I \cap (d_{i, \alpha} \times I)_{\gamma_\alpha} \neq \emptyset$ , then  $\gamma \geq \gamma_\alpha$ . In this case, each component of  $e_\gamma^l \times I \cap e_{\gamma_\alpha}^l \times I$  is  $(c \times I)_\gamma \subset e_{\gamma_\alpha}^l \times (0, 1)$  when  $\gamma > \gamma_\alpha \in m(k)$ . Hence (2) holds. Q.E.D.

**Lemma 3.4.7.** (1) Suppose that  $\alpha < \lambda < \beta$ . Then  $\delta(\alpha, \beta) \geq \delta(\alpha, \lambda), \delta(\lambda, \beta)$  and  $\theta(\alpha, \beta) \leq \theta(\alpha, \lambda), \theta(\lambda, \beta)$ .

(2)  $D_{\alpha, \beta}$  is a disk in  $F^l$ .

(3) For  $\alpha \leq \lambda \leq \beta$ , let  $h_{\alpha, \beta}^\lambda = \cup_{i=\delta_0+1}^{\theta_0-1} d_{i, \lambda} \cup_{i=\delta_0}^{\theta_0} e_{i, \lambda}$  where  $\theta_0 = \theta(\alpha, \beta)$  and  $\delta_0 = \delta(\alpha, \beta)$ .

(i) if  $d_j^l \cap D(\alpha, \beta) \neq \emptyset$ , then  $d_j^l \cap D(\alpha, \beta) \subset \cup_\lambda (h_{\alpha, \beta}^\lambda \times I)_{\gamma_\lambda}$ . Furthermore, if  $j \neq j_\lambda$  for each  $\alpha \leq \lambda \leq \beta$ , then  $j > \gamma_\lambda$  for some  $\alpha \leq \lambda \leq \beta$ ,

(ii) If  $\gamma > \gamma_{i, \alpha}$  for  $\delta(\alpha, \beta) \leq i \leq \theta(\alpha, \beta)$  and  $e_\gamma^l \times I \cap D(\alpha, \beta) \neq \emptyset$ , then  $e_\gamma^l \times I \cap D(\alpha, \beta) \subset \cup_\lambda (h_{\alpha, \beta}^\lambda \times I)_{\gamma_\lambda}$ . Furthermore,  $\gamma \geq \gamma_\lambda$  for some  $\alpha \leq \lambda \leq \beta$ .

**Proof.** By Lemma 3.3.1, if  $i \neq r$ , then  $f_{i, \lambda} \neq f_{r, \lambda}$ . Hence  $D_{\lambda, \lambda+1, i} \cap D_{\lambda, \lambda+1, r} = \emptyset$  if  $i \neq r$ . By Lemmas 3.4.4 and 3.4.5,  $D_{\lambda, \lambda+1}$  is a disk in  $F^l$ . See in Definition 3.4.1.

We first prove that  $\delta(\alpha, \alpha+2) \geq \delta(\alpha, \alpha+1), \delta(\alpha+1, \alpha+2)$  and  $\theta(\alpha, \alpha+2) \leq \theta(\alpha, \alpha+1), \theta(\alpha+1, \alpha+2)$ .

Suppose, otherwise, that  $\delta(\alpha, \alpha+2) < \delta = \delta(\alpha, \alpha+1)$ . Then  $f_{i, \alpha} = f_{i, \alpha+1} = f_{i, \alpha+2}$  and  $\gamma_{i, \alpha} = \gamma_{i, \alpha+1} = \gamma_{i, \alpha+2}$  for  $\delta \leq i \leq -1$ . Furthermore,  $\gamma_{i, \alpha} < j_\alpha, j_{\alpha+1}, j_{\alpha+2}$  for  $\delta \leq i \leq -1$ .

Now let  $D_1 = \cup_{i=\delta+1}^0 D_{\alpha,\alpha+1,i} \cup_{i=\delta}^{-1} A_{\alpha,\alpha+1,i}$ ,  $D_2 = \cup_{i=\delta+1}^0 D_{\alpha+1,\alpha+2,i} \cup_{i=\delta}^{-1} A_{\alpha+1,\alpha+2,i}$ . Then  $D_1 \subset D_{\alpha,\alpha+1}$  and  $D_2 \subset D_{\alpha+1,\alpha+2}$  are two disks.

Now by Lemma 3.2.2 and Lemma 3.4.2,  $d_{i,\alpha+2}$  is disjoint from  $D_{\alpha,\alpha+1,i}$ , otherwise,  $d_{0,\alpha+2} \subset D_{\alpha,\alpha+1,0}$ . Hence  $D_{\alpha,\alpha+1,i} \cap D_{\alpha+1,\alpha+2,i} = d_{i,\alpha+1}$ . Since  $\gamma_{i,\alpha} < j_\alpha, j_{\alpha+1}, j_{\alpha+2}$ , by Lemma 3.4.3,  $A_{\alpha+1,\alpha+2,i}$  is disjoint from  $\text{int}D_{\alpha,\alpha+1,r}$ . Similarly,  $A_{\alpha,\alpha+1,i}$  is disjoint from  $\text{int}D_{\alpha+1,\alpha+2,r}$ . By the proof of Lemma 3.4.4,  $A_{\alpha,\alpha+1,i}$  is disjoint from  $A_{\alpha+1,\alpha+2,r}$  for  $i \neq r$ . Note that  $A_{\alpha,\alpha+1,i} \cap A_{\alpha+1,\alpha+2,i} = e_{i,\alpha+1}$ ; Otherwise,  $e_{i,\alpha+2}$  separates  $e_{i,\alpha}, e_{i,\alpha+1}$  in  $e_{\gamma_{i,\alpha}}^l \times I$ , and  $d_{i+1,\alpha+2}$  separates  $d_{i+1,\alpha}$  and  $d_{i+1,\alpha+1}$  in  $E_{f_{i+1,\alpha}}^l$ , but by the proof of Lemma 3.4.2, this is impossible. Now  $D_1 \cup D_2$  is a disk in  $F^l$ . Hence  $d_{\delta,\alpha+1}$  separates  $d_{\delta,\alpha}$  and  $d_{\delta,\alpha+2}$  in  $E_{f_{\delta,\alpha}}^l$ . Since  $F^l$  is generated by  $\cup_f E_f^l \cup_\gamma e_\gamma^l$ . So if  $(\partial_1 e_\gamma^l) \times I \subset E_f^l$  for some  $f$ , then  $(\partial_2 e_\gamma^l) \times I$  is disjoint from  $E_f^l$ . Now if  $\gamma_{\delta-1,\alpha} = \gamma_{\delta-1,\alpha+2}$ , then  $\gamma_{\delta-1,\alpha} = \gamma_{\delta-1,\alpha+1}$ . This means that  $\delta(\alpha, \alpha+2) \geq \delta = \delta(\alpha, \alpha+1), \delta(\alpha+1, \alpha+2)$ , a contradiction.

Similarly,  $\theta_0 = \theta(\alpha, \alpha+2) \leq \theta(\alpha, \alpha+1), \theta(\alpha+1, \alpha+2)$ .

Let  $\delta_0 = \delta(\alpha, \alpha+2)$  and  $\theta_0 = \theta(\alpha, \alpha+2)$ . Let  $D_* = \cup_{i=\delta_0+1}^{\theta_0-1} D_{\alpha,\alpha+1,i} \cup_{i=\delta_0}^{\theta_0} A_{\alpha,\alpha+1,i}$ ,  $D_{**} = \cup_{i=\delta_0+1}^{\theta_0-1} D_{\alpha+1,\alpha+2,i} \cup_{i=\delta_0}^{\theta_0} A_{\alpha+1,\alpha+2,i}$ . Then  $D_* \subset D_{\alpha,\alpha+1}$  and  $D_{**} \subset D_{\alpha+1,\alpha+2}$ .

By the above argument,  $D(\alpha, \alpha+2) = D_* \cup D_{**}$  is a disk in  $F^l$ . By Lemma 3.4.2, Lemma 3.4.3 and Lemma 3.4.6, (3) and (4) hold.

By induction,  $\alpha < \lambda < \beta$ . Then  $\delta(\alpha, \beta) \geq \delta(\alpha, \lambda), \delta(\lambda, \beta)$  and  $\theta(\alpha, \beta) \leq \theta(\alpha, \lambda), \theta(\lambda, \beta)$ . Furthermore,  $D_{\alpha,\beta} = \cup_{\lambda=\alpha}^{\beta-1} (\cup_{i=\delta(\alpha,\beta)+1}^{\theta(\alpha,\beta)-1} D_{\lambda,\lambda+1,i} \cup_{i=\delta(\alpha,\beta)}^{\theta(\alpha,\beta)} A_{\lambda,\lambda+1,i})$  is a disk as in Figure 10(a). (3) is immediately from Lemmas 3.4.2, 3.4.3 and 3.4.6. Q.E.D.

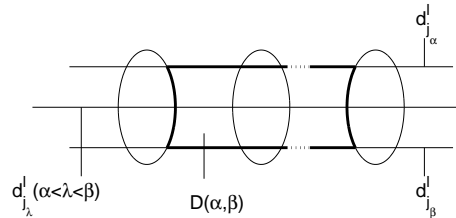


Figure 10(a)

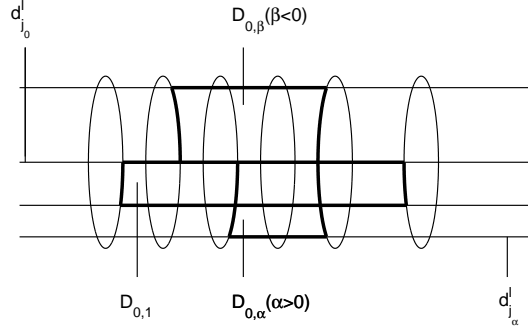


Figure 10(b)

**Lemma 3.4.8.**  $\cup_{\alpha \neq 0} D_{0,\alpha}$  is a disk.

**Proof.** By Lemma 3.4.7,  $\delta(0, \alpha) \leq \delta(0, \beta) \leq \theta(0, \beta) \leq \theta(0, \alpha)$  if  $0 < \alpha < \beta$  or  $\beta < \alpha < 0$ . Now suppose that  $L(c_{l+1}^l) = \{j_{\beta_1}, \dots, j_{-1}, j_0, j_1, \dots, j_{\beta_2}\}$ . Then

$$\cup_{\alpha \neq 0} D_{0,\alpha} = \cup_{\alpha=\beta_1}^{\beta_2-1} (\cup_{i=\delta(0,\alpha)+1}^{\theta(0,\alpha)-1} D_{\alpha,\alpha+1,i} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} A_{\alpha,\alpha+1,i}).$$

Since  $j_0 < j_\alpha$ ,  $\gamma_{i,0} < j_0 < j_\alpha$  for  $\delta(0) \leq i \leq \theta(0)$  and  $\alpha \neq 0$ .

Now if  $\beta \neq \alpha, \alpha + 1$ , then  $\text{int} D_{\beta,\beta+1,i}$  is disjoint from  $D_{\alpha,\alpha+1,i}$ . Otherwise, one of  $d_{i,\beta}$  and  $d_{i,\beta+1}$ , say  $d_{i,\beta}$ , separates  $d_{i,\alpha}$  and  $d_{i,\alpha+1}$  in  $E_{f_{i,\alpha}}^l$ , but by the proof of Lemma 3.4.2,  $d_{0,\beta}$  separates  $d_{0,\alpha}$  and  $d_{0,\alpha+1}$  in  $E_0^l$ , a contradiction. Similarly, if  $\{\beta, \beta+1\} \cap \{\alpha, \alpha+1\} = \emptyset$ , then  $A_{\beta,\beta+1,i}$  is disjoint from  $A_{\alpha,\alpha+1,i}$ . If  $\{\beta, \beta+1\} \cap \{\alpha, \alpha+1\} = \{\beta\}$ , then  $A_{\beta,\beta+1,i} \cap A_{\alpha,\alpha+1,i} = e_{i,\beta}$ .

Since  $\gamma_{i,\alpha} = \gamma_{i,0} < j_0$  for each  $\alpha$  and  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$ , by the proof of Lemma 3.4.4,  $\text{int} A_{\alpha,\alpha+1,i}$  is disjoint from  $A_{\beta,\beta+1,r}$ . By Lemma 3.4.3,  $\text{int} A_{\alpha,\alpha+1,i}$  is disjoint from  $D_{\beta,\beta+1,r}$ . Now  $\cup_{\alpha \neq 0} D_{0,\alpha}$  is a disk as in Figure 10(b). Q.E.D.

### 3.5 $a_\alpha$ and $a_\alpha^0$

For each  $j_\alpha \in L(c_{l+1}^l)$ , recalling the equality:  $d_{j_\alpha}^l = d_{0,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$ .

**Definition 3.5.1.** For  $\alpha \neq 0$ ,

- (1) let  $a_\alpha = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$ ,
- (2) let  $a_\alpha^0 = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,0} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,0}$ .

**Lemma 3.5.2.** (1)  $a_\alpha \subset d_{j_\alpha}^l$  and  $a_\alpha^0 \subset d_{j_0}^l$ .

(2) If  $0 < \alpha < \beta$ , then  $a_\beta^0 \subset a_\alpha^0$ .

(3) If  $\beta < \alpha < 0$ , then  $a_\beta^0 \subset a_\alpha^0$ .

**Proof.** The lemma follows from Definition 3.5.1 and Lemma 3.4.7. Q.E.D.

**Lemma 3.5.3.** (1) Suppose that  $s(w_{\gamma_\alpha}) = +$  or  $j_\alpha \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ . If  $e_\gamma^l \times I \cap (inta_\alpha \times I)_{\gamma_\alpha} \neq \emptyset$ , then  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

(2) Suppose that  $s(w_{\gamma_\alpha}) = -$ . If  $e_\gamma^l \times I \cap (inta_\alpha \times I)_{\gamma_\alpha} \neq \emptyset$ , then either  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$  or  $\gamma \geq \gamma_\alpha$ .

**Proof.** By Definition 3.5.1,  $a_\alpha = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$ . By Lemma 3.3.4,  $\gamma_{i,\alpha} = \gamma_{i,0}$ ,  $f_{i,\alpha} = f_{i,0}$  for  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$ .

(1) Suppose that  $s(w_{\gamma_\alpha}) = +$  or  $j_\alpha \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ . In this case, by Definition 3.2.1,  $(inta_\alpha \times I)_{\gamma_\alpha} = inta_\alpha$ . If  $\gamma > j_\alpha$ , then, by Proposition 4(3),  $a_\alpha$  is disjoint from  $e_\gamma^l \times I$ . So  $j_\alpha > \gamma$ . By Lemma 3.3.1,  $intd_{i,\alpha}$  is disjoint from  $e_\gamma^l \times I$ . Note that  $\gamma_{i,\alpha} = \gamma_{i,0}$  for  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$ . Now by the proof of Lemma 3.2.3,  $inta_\alpha$  is disjoint from  $e_\gamma^l \times I$  for  $\gamma > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

(2) Suppose that  $s(w_{\gamma_\alpha}) = -$ ,  $\gamma_\alpha > \gamma > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$  and  $e_\gamma^l \times I \cap (inta_\alpha \times I)_{\gamma_\alpha} \neq \emptyset$ .

If  $j_\alpha > \gamma$ , then, by (1),  $inta_\alpha$  is disjoint from  $e_\gamma^l \times I$ . Now suppose that  $j_\alpha < \gamma$ . Since  $j_\alpha \in I(w_{\gamma_\alpha}, l)$ , by Lemma 2.2.5,  $j_\alpha \notin I(w_\gamma, l)$ . By Proposition 4(3),  $a_\alpha \subset d_{j_\alpha}^l$  is disjoint from  $e_\gamma^l \times I$ .

Since  $\gamma_\alpha > \gamma$ , each component of  $e_\gamma^l \times I \cap e_{\gamma_\alpha}^l \times I$  is  $(c \times I)_{\gamma_\alpha}$  where  $c \subset inte_{\gamma_\alpha}^l$ . Now if  $e_\gamma^l \times I \cap (inta_\alpha \times I)_{\gamma_\alpha} \neq \emptyset$ , then  $inta_\alpha \cap e_{\gamma_\alpha}^l \times I \neq \emptyset$ , a contradiction. Q.E.D.

**Lemma 3.5.4.** If  $\alpha, \beta \neq 0$  and  $\alpha \neq \beta$ , then  $(a_\alpha \times I)_{\gamma_\alpha}$  is disjoint from  $(a_\beta \times I)_{\gamma_\beta}$ .

**Proof.** There are three cases:

Case 1.  $j_\alpha, j_\beta \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$  with  $s(w_\gamma) = -$ .

Now by Definition 3.2.1,  $(a_\alpha \times I)_{\gamma_\alpha} = a_\alpha \subset d_{j_\alpha}^l$  and  $(a_\beta \times I)_{\gamma_\beta} = a_\beta \subset d_{j_\beta}^l$ . Since  $j_\alpha \neq j_\beta$ , the lemma holds.

Case 2.  $j_\alpha \notin I(w_\gamma, l)$  for each  $\gamma \in m(k)$  with  $s(w_\gamma) = -$ , and  $j_\beta \in I(w_{\gamma_\beta}, l)$  for some  $\gamma_\beta \in m(l)$  with  $s(w_{\gamma_\beta}) = -$ .

Since  $\gamma_\beta > j_\beta$ ,  $\gamma_\beta > j_0 = m^{l+1}$ . Since  $\gamma_{i,\alpha} = \gamma_{i,0}$  for  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$ . Hence  $\gamma_\beta > \gamma_{i,0}$  for  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$ , and  $\gamma_\beta > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . By the definition,  $a_\alpha = \bigcup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \bigcup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$  and  $a_\beta = \bigcup_{i=\delta(0,\beta)}^{\theta(0,\beta)} d_{i,\beta} \bigcup_{i=\delta(0,\beta)}^{\theta(0,\beta)} e_{i,\beta}$ . Now if  $j_\alpha < \gamma_\beta$ , then, by Proposition 4(3),  $a_\alpha$  is disjoint from  $(a_\beta \times I)_{\gamma_\beta}$ .

Suppose that  $j_\alpha > \gamma_\beta$ . By Proposition 4(2), either  $d_{j_\alpha}^l \cap e_{\gamma_\beta}^l \times I = \emptyset$  or each component of  $d_{j_\alpha}^l \cap e_{\gamma_\beta}^l \times I$  is a core of  $e_{\gamma_\beta}^l \times I$ . Hence either  $d_{j_\alpha}^l \cap (a_\beta \times I)_{\gamma_\beta} = \emptyset$  or each component of  $d_{j_\alpha}^l \cap (a_\beta \times I)_{\gamma_\beta}$  is a core of  $(a_\beta \times I)_{\gamma_\beta}$ . By Lemma 3.2.2,  $d_{\delta(0,\alpha),\alpha} \cup d_{\theta(0,\alpha),\alpha}$  is disjoint from  $(d_{\delta(0,\beta),\beta} \times I)_{\gamma_\beta} \cup (d_{\theta(0,\beta),\beta} \times I)_{\gamma_\beta}$ . Now if  $a_\alpha \cap (a_\beta \times I)_{\gamma_\beta} \neq \emptyset$ , then  $\text{inta}_\alpha \cap (a_\beta \times I)_{\gamma_\beta} \neq \emptyset$ . By Lemma 3.5.3(1), this is impossible.

Case 3.  $s(w_{\gamma_\alpha}) = -$ ,  $s(w_{\gamma_\beta}) = -$ .

Now if  $\gamma_\alpha = \gamma_\beta$ , then the lemma holds.

Suppose that  $\gamma_\alpha < \gamma_\beta$ . By Proposition 4(1), each component of  $e_{\gamma_\beta}^l \times I \cap e_{\gamma_\alpha}^l \times I$  is  $(c \times I)_{\gamma_\beta}$  where  $c \subset e_{\gamma_\alpha}^l$  is a core of  $e_{\gamma_\alpha}^l \times (0, 1)$ . By (2),  $a_\beta$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$ . Hence the lemma holds. **Q.E.D.**

**Lemma 3.5.5.** For each  $\alpha \neq 0$ , there are an arc  $a_{0,\alpha}$  in  $E_{f_{\delta(0,\alpha),\alpha}}^l$  and an arc  $b_{0,\alpha}$  in  $E_{f_{\theta(0,\alpha),\alpha}}^l$  such that

- (1)  $\partial_1 a_{0,\alpha} = \partial_1 a_\alpha$  and  $\partial_2 a_{0,\alpha} = \partial_1 a_\alpha^0$ ,  $\partial_1 b_{0,\alpha} = \partial_2 a_\alpha$  and  $\partial_2 b_{0,\alpha} = \partial_2 a_\alpha^0$ ,
- (2)  $a_{0,\alpha} \cup a_\alpha \cup b_{0,\alpha} \cup a_\alpha^0$  bounds a disk  $D_{0,\alpha}^*$  in  $F^l$ .
- (3)  $d_j^l \cap a_{0,\alpha} \subset \bigcup_{\lambda=0}^\alpha (d_{\delta(0,\alpha),\lambda} \times I)_{\gamma_\lambda}$ ,  $d_j^l \cap b_{0,\alpha} \subset \bigcup_{\lambda=0}^\alpha (d_{\theta(0,\alpha),\lambda} \times I)_{\gamma_\lambda}$ .
- (4)  $e_\gamma^l \times I \cap a_{0,\alpha} \subset \bigcup_{\lambda=0}^\alpha (d_{\delta(0,\alpha),\lambda} \times I)_{\gamma_\lambda}$  and  $e_\gamma^l \times I \cap b_{0,\alpha} \subset \bigcup_{\lambda=0}^\alpha (d_{\theta(0,\alpha),\lambda} \times I)_{\gamma_\lambda}$ .

Furthermore, if  $e_\gamma^l \times I \cap (\text{inta}_{0,\alpha} \cup \text{intb}_{0,\alpha}) \neq \emptyset$ , then  $\gamma \geq \gamma_\lambda$  for some  $\alpha \leq \lambda \leq \beta$ .

**Proof.** We may assume that  $\alpha > 0$ . By Lemma 3.4.7, for  $0 \leq \lambda \leq \alpha$ ,  $\delta(0, \alpha) \leq \delta(0, \lambda) \leq \theta(0, \lambda) \leq \theta(0, \alpha)$ . By Lemma 3.4.7(3) and Definition 3.5.1,  $h_{0,\alpha}^\alpha \cup d_{\delta(0,\alpha),\alpha} \cup d_{\theta(0,\alpha),\alpha} = a_\alpha$ . By Lemma 3.4.7 and Lemma 3.5.4,  $d_{\delta(0,\alpha),\lambda} \cap D_{0,\alpha} = \partial_2 d_{\delta(0,\alpha),\lambda}$  for  $0 \leq \lambda \leq \alpha$ .

Now let  $a_{0,\alpha}$  be in  $E_{f_{\delta(0,\alpha)}}^l$  such that  $\partial_1 a_{0,\alpha} = \partial_1 d_{\delta(0,\alpha),\alpha} = \partial_1 a_\alpha$  and  $\partial_2 a_{0,\alpha} = \partial_1 d_{\delta(0,\alpha),0} = \partial_1 a_\alpha^0$  as in Figure 11(a). In fact,  $a_{0,\alpha}$  can be obtained by pushing  $d_{\delta(0,\alpha),0} \cup a^* \cup d_{\delta(0,\alpha),\alpha}$  slightly where  $a^* \subset \partial D_{0,\alpha} \cap E_{f_{\delta(0,\alpha),0}}^l$  as in Figure 11(a). Then  $a_{0,\alpha}$  is disjoint from  $D_{0,\alpha}$ . Similarly, let  $b_{0,\alpha}$  be in  $E_{f_{\theta(0,\alpha),0}}^l$  such that  $\partial_1 b_{0,\alpha} = \partial_2 d_{\theta(0,\alpha),\alpha} = \partial_2 a_\alpha$  and  $\partial_2 b_{0,\alpha} = \partial_2 d_{\theta(0,\alpha),0} = \partial_2 a_\alpha^0$  as

in Figure 11(a). Hence  $a_{0,\alpha} \cup a_\alpha \cup b_{0,\alpha} \cup b_\alpha$  bounds a disk  $D_{0,\alpha}^*$  in  $F^l$ .

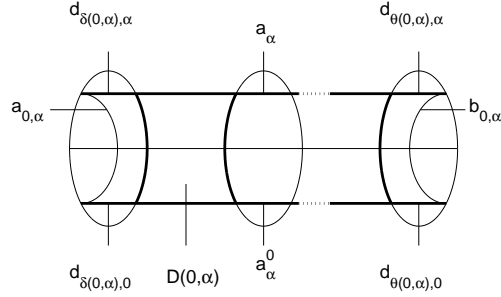


Figure 11(a)

Suppose that  $d_j^l \cap a_{0,\alpha} \neq \emptyset$ . Then, by Lemma 3.1.6 and Lemma 3.2.2, one component of  $d_j^l \cap E_{f_{\delta(0,\alpha),\alpha}}^l$ , say  $c$ , lies in  $D_{0,\alpha,\delta(0,\alpha)}$ . If  $\partial c$  is disjoint from  $A_{0,\alpha,\delta(0,\alpha)}$ , then  $a_{0,\alpha}$  can be isotoped to be disjoint from  $c$ . If one end point of  $c$  lies in  $A_{0,\alpha,\delta(0,\alpha)}$ , then, by the proof of Lemma 3.4.1, (2) holds. In this case, either  $j = j_\lambda$  or  $j > \gamma_\lambda$  for some  $\alpha \leq \lambda \leq \beta$ .

(3) follows from Lemma 3.2.4 and the proof of Lemma 3.4.3. Q.E.D.

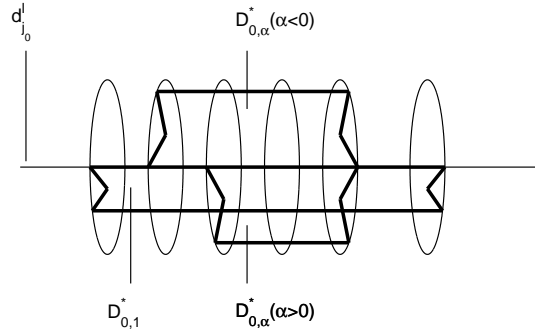


Figure 11(b)

**Lemma 3.5.6.**  $\cup_{\alpha \neq 0} D_{0,\alpha}^*$  is a disk.

**Proof.** By Lemma 3.4.8,  $\cup_{\alpha \neq 0} D_{0,\alpha}$  is a disk. By Lemma 3.5.4, Lemma 3.4.7,  $d_{\delta(0,\beta),\beta} \cap (\cup_{\alpha \neq 0} D_{0,\alpha}) = \partial_2 d_{\delta(0,\beta),\beta}$  for  $\beta \neq 0$  as in Figure 10(b). We may assume that  $\delta(0, -1) \geq \delta(0, 1)$ . Then, by Lemma 3.4.7,  $\delta(0, \alpha) \geq \delta(0, 1)$  for each  $\alpha$ . Then  $d_{\delta(0,1),0} \cap (\cup_{\alpha \neq 0} D_{0,\alpha}) = \partial_2 d_{\delta(0,1),0}$ . Furthermore,  $d_{\delta(0,\alpha),0} \subset a_1^0$  for each  $\alpha$ . By the proof of Lemma 3.5.5,  $\cup_{\alpha \neq 0} D_{0,\alpha}^*$  is a disk as in Figure 11(b). Q.E.D.



**Lemma 3.5.7.** (1) For each  $\alpha \neq 0$ ,  $d_{j_0}^l \cap D_{0,\alpha}^* = a_\alpha^0$ .

(2) If  $r < j_0$ , then  $d_r^l$  is disjoint from  $D_{0,\alpha}^*$ .

**Proof.** By Lemma 3.5.4,  $a_\alpha^0 \subset \partial D_{0,\alpha}^*$ . Since  $j_0 < j_\alpha$  for each  $\alpha \neq 0$ . Hence, by Proposition 4(3),  $d_{j_0}^l$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$  for  $\alpha \neq 0$ . By Lemma 3.5.5(3) and Lemma 3.4.7,  $d_{j_0}^l - a_\alpha^0$  is disjoint from  $D_{0,\alpha}^*$ .

Suppose that  $r < j_0$ . Then  $r < j_\alpha$  for each  $\alpha$ . Hence  $d_r^l$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$ . By Lemma 3.5.5(3) and Lemma 3.4.7, (2) holds. Q.E.D.

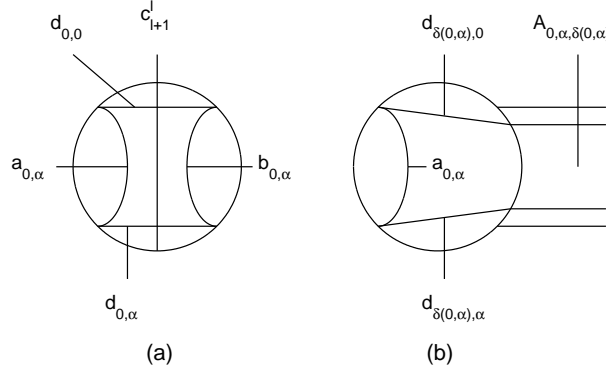


Figure 12

**Lemma 3.5.8.** (1) If  $\delta(0, \alpha) = \theta(0, \alpha) = 0$ ,  $c_i^l$  intersects  $a_{0,\alpha} \cup b_{0,\alpha}$  in one point if and only if  $c_i^l$  intersects  $d_{0,\alpha} \cup d_{0,0}$  in one point.

(2) If  $\delta(0, \alpha) \neq \theta(0, \alpha)$ , then  $c_i^l$  intersects  $a_{0,\alpha}$  in a one point if and only if  $c_i^l$  intersects  $d_{\delta(0,\alpha),\alpha} \cup d_{\delta(0,\alpha),0}$  in one point,  $c_i^l$  intersects  $b_{0,\alpha}$  in one point if and only if  $c_i^l$  intersects  $d_{\theta(0,\alpha),\alpha} \cup d_{\theta(0,\alpha),0}$  in one point.

**Proof.** (1) Since  $\delta(0, \alpha) = \theta(0, \alpha) = 0$ ,  $a_\alpha = d_{0,\alpha}$  and  $a_\alpha^0 = d_{0,0}$ . Hence  $a_{0,\alpha} \subset E_0^l$  and  $b_{0,\alpha} \subset E_0^l$  as in Figure 12(a). By Lemma 3.1.2,  $\partial d_{0,0} \cup \partial d_{0,\alpha} \subset \partial d_{j_0}^l \cup \partial d_{j_\alpha}^l \cup (\partial e_\gamma^l) \times I$ . By Proposition 4(4),(5),  $d_{0,0} \cup d_{0,\alpha}$  is disjoint from  $\partial c_i^l$ , and  $c_i^l$  is properly embedded in  $E_f^l$ . Since  $j_0, j_\alpha \in L(c_{l+1}^l)$ ,  $c_{l+1}^l$  intersects each of  $d_{0,0}$  and  $d_{0,\alpha}$  in one point, we can move  $c_{l+1}^l$  so that  $a_{0,\alpha} \cup b_{0,\alpha}$  is disjoint from  $c_{l+1}^l$ . Similarly, if  $c_i^l$  intersects each of  $d_{0,0}$  and  $d_{0,\alpha}$  in one point, then  $c_i^l$  is disjoint from  $a_{0,\alpha} \cup b_{0,\alpha}$ . Hence (1) holds.

(2) By Lemma 3.3.1,  $d_{\delta(0,\alpha),0}$  and  $d_{\delta(0,\alpha),\alpha}$  are properly embedded arcs in  $E_{f_{\delta(0,\alpha),\alpha}}^l$  and  $d_{\theta(0,\alpha),0}$  and  $d_{\theta(0,\alpha),\alpha}$  are properly embedded arcs in  $E_{f_{\theta(0,\alpha),\alpha}}^l$ . Since  $\delta(0, \alpha) \neq \theta(0, \alpha)$ , by

Lemma 3.3.1,  $f_{\delta(0,\alpha),\alpha} \neq f_{\theta(0,\alpha),\alpha}$ . By Proposition 4(5),  $\partial c_i^l$  is disjoint from  $e_\gamma^l \times I$  and  $d_j^l$ . Hence  $c_i^l$  is disjoint from  $A_{0,\alpha,\delta(0,\alpha)} \cup A_{0,\alpha,\theta(0,\alpha)}$ . Note that  $\partial_2 d_{\delta(0,\alpha),0} \cup \partial_2 d_{\delta(0,\alpha),\alpha} \subset A_{0,\alpha,\delta(0,\alpha)}$  and  $\partial_1 d_{\theta(0,\alpha),0} \cup \partial_1 d_{\theta(0,\alpha),\alpha} \subset A_{0,\alpha,\theta(0,\alpha)}$  as in Figure 12(b). By the argument in (1), (2) holds. Q.E.D.

**Remark 3.5.9.** Since Propositions 1-6 hold for  $k \leq l$ . By Lemma 3.1.5,  $m^k = \text{Min}L(v_{k+1}^k)$  for  $k \leq l$ . By Propositions 1 and 2, all the arguments in Sections 3.1-3.5 are true when we take place of  $c_i^l, d_j^l, e_\gamma^l, F^l$  with  $v_i^l, w_j^l, b_\gamma^l, P^l$ ,  $s(w_\gamma) = -$  with  $s(w_\gamma) = +$  in Lemma 3.1.6 and Lemma 3.2.3,  $s(w_\gamma) = s(w_{\gamma_j}) = -$  with  $s(w_\gamma) = s(w_{\gamma_j}) = +$  in Lemma 3.2.4.

## 4 The Proofs of Propositions 4-6 for the case: $k = l + 1$ and $s(v_{l+1}) = +$

Assume now that Propositions 1-6 hold for each  $0 \leq k \leq l$ . Now we only need to prove that Propositions 1-6 hold for  $k = l + 1$ .

There are two cases:

1.  $s(v_{l+1}) = +$ .
2.  $s(v_{l+1}) = -$ .

In this chapter, we shall prove that Propositions 4-6 hold for Case 1. Hence, in this chapter, we assume that  $s(v_{l+1}) = +$ . We first construct  $c_i^{l+1}, d_j^{l+1}, e_\gamma^{l+1} \times I, F^{l+1}$  from  $c_i^l, d_j^l, e_\gamma^l \times I, F^l$ , then we shall prove Propositions 4-6 for this case.

### 4.1 The element of the construction

Since Propositions 1-6 hold for  $k \leq l$ . So  $L(c_i^l)$ ,  $L(d_j^l)$ , and  $m(k)$  are well defined for  $k \leq l$ . Recall that  $m^{l+1}$  defined in Definition 3.3.0.

**Definition 4.1.1.** Let  $m(l+1) = m(l) \cup \{m^{l+1}\}$ . In particular, if  $m^{l+1} = \emptyset$ , then let  $m(l+1) = m(l)$ .

Without loss of generality, we assume that  $c_{l+1}^l \subset E_0^l$ . Recalling the argument in Section 3.3, we can number all the elements in  $L(c_{l+1}^l)$  as  $\dots, j_{-1}, j_0 = m^{l+1}, j_1, \dots$  according to the order of  $(\cup_{j \in L(c_{l+1}^l)} d_{j,0}^l) \cap c_{l+1}^l$  lying in  $c_{l+1}^l$ . In some time, we shall use  $j_0$  to take place of  $m^{l+1}$ . By Lemma 3.3.1 and Remark 3.3.2, for each  $j_\alpha$ ,  $d_{j_\alpha}^l = \cup_{i=\delta(\alpha)}^{\theta(\alpha)} d_{i,\alpha} \cup_{i=\delta(\alpha)}^{\theta(\alpha)} e_{i,\alpha}$ .

**Lemma 4.1.2.** Suppose that  $j_0 = m^{l+1} \neq \emptyset$ .

(1) In  $F^l$ ,  $c_{l+1}^l$  intersects  $d_{j_0}^l$  in one point. Furthermore, if  $r < j_0$ , then  $d_r^l$  is disjoint from  $c_{l+1}^l$ ; if  $\gamma < j_0$ , then  $e_\gamma^l \times I$  is disjoint from  $c_{l+1}^l$ .

(2) In  $P^l$ ,  $v_{l+1}^l$  intersects  $w_{j_0}^l$  in one point. Furthermore, if  $r < j_0$ , then  $v_{l+1}^l$  is disjoint from  $w_r^l$ ; if  $\gamma < j_0$ , then  $b_\gamma^l \times I$  is disjoint from  $v_{l+1}^l$ .

**Proof.** By assumptions, Propositions 1-6 hold for  $k = l$ .

(1) By Proposition 4(4),  $c_{l+1}^l$  is a properly embedded arc in  $F^l$  lying in  $E_0^l$ . Since  $j_0 = m^{l+1} \in L(c_{l+1}^l)$ ,  $d_{0,0}$  intersects  $c_{l+1}^l$  in one point. By Proposition 4(5) and Lemma 3.1.2,  $d_{0,0} \cap c_{l+1}^l = \text{int}c_{l+1}^l \cap \text{int}d_{0,0}$ . Suppose, otherwise,  $d_{j_0}^l$  intersects  $c_{l+1}^l$  in at least two points. Then, by Lemma 3.1.6,  $r \in L(c_{l+1}^l)$  for some  $r < j_0$ , contradicting the minimality of  $j_0$ . Similarly, if  $r < j_0$ , then  $c_{l+1}^l$  is disjoint from  $d_r^l$ . By Lemma 3.2.4, if  $\gamma < j_0$ , then  $e_\gamma^l \times I$  is disjoint from  $c_{l+1}^l$ .

(2) follows from Remark 3.5.9 and Propositions 1 and 2. Q.E.D.

**Lemma 4.1.3.** (1) If  $j_0 = m^{l+1} \neq \emptyset$ , then  $s(w_{j_0}) = -$ .

(2)  $W_{j_0}^l$  is a properly embedded disk in  $\mathcal{W}_-$  such that  $\partial W_{j_0}^l$  intersects  $c_{l+1}^l$  in one point which lies in  $\text{int}d_{0,0}$ .

**Proof.** (1) Now consider  $P^l$ . By assumptions, Propositions 1-3 hold for  $k = l$ . Suppose, otherwise,  $s(w_{j_0}) = +$ . By assumption,  $s(v_{l+1}) = +$ . By Proposition 3,  $V_{l+1}^l$  is a properly embedded disk in  $\mathcal{V}_+$  and  $W_{j_0}^l$  is a properly embedded disk in  $\mathcal{W}_+$ . By Definition 2.3.1 and Lemma 2.2.4,  $I(v_{l+1}, l) = I(v_{l+1}) - \{1, \dots, l\} = \emptyset$ . By Lemma 3.1.5,  $m^{l+1} = \text{Min}L(v_{l+1}^l)$ . Note that  $j_0 = m^{l+1}$ . By Proposition 3,  $V_{l+1}^l \cap W_{j_0}^l = (v_{l+1}^l \cup_{r \in I(v_{l+1}, l)} v_r^l) \cap (w_{j_0}^l \cup_{r \in I(w_{j_0}, l)} w_r^l) = v_{l+1}^l \cap (w_{j_0}^l \cup_{r \in I(w_{j_0}, l)} w_r^l)$ . If  $r \in I(w_{j_0}, l)$ , then, by Lemma 2.2.4,  $r < j_0$ . By Lemma 4.1.2(2),  $w_r^l$  is disjoint from  $v_{l+1}^l$ . By also Lemma 4.1.2(2),  $V_{l+1}^l$  intersects  $W_{j_0}^l$  in only one point. Hence  $\mathcal{V}_+ \cup \mathcal{W}_+$  is stabilized, a contradiction.

(2) By (1) and Proposition 6,  $W_{j_0}^l$  is a properly embedded disk in  $\mathcal{W}_-$ . By Proposition 6,  $W_{j_0}^l \cap c_{l+1}^l = (d_{j_0}^l \cup_{r \in I(w_{j_0}, l)} d_r^l) \cap c_{l+1}^l$ . Since  $r \in I(w_{j_0}, l)$ ,  $r < j_0$ . By Lemma 4.1.2,  $\partial W_{j_0}^l$  intersects  $c_{l+1}^l$  in one point lying in  $\text{int}d_{0,0}$ . Q.E.D.

**Lemma 4.1.4.** Suppose that  $j_0 = m^{l+1} \neq \emptyset$ .

(1)  $j_0 \notin I(w_\gamma, l)$  for each  $\gamma \in m(l) = \{m^1, \dots, m^l\}$  with  $s(w_\gamma) = -$ . Furthermore, if  $r \in I(w_{j_0}, l)$ , then  $r \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ .

(2)  $d_{j_0}^l$  is properly embedded in  $F^l$ ,  $d_r^l$  is properly embedded in  $F^l$  for  $r \in I(w_{j_0}, l)$ .

**Proof.** By Lemma 4.1.3,  $s(w_{j_0}) = -$ . Now if  $j_0 \in I(w_\gamma, l)$  for some  $\gamma \in m(l)$ , then, by Lemma 2.2.5,  $s(w_\gamma) = +$ . By Lemma 3.1.1,  $d_{j_0}^l$  is regular in  $F^l$ . By Definition 2.1.5,  $d_{j_0}^l$  is properly embedded in  $F^l$ .

Suppose that  $r \in I(w_{j_0}, l)$ . By Definition 2.3.3,  $j_0 \in \{1, \dots, n\} - m(l)$ . Hence  $j_0 \notin m(l)$ . By Lemma 2.2.5,  $r \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ . By Lemma 3.1.1 and Definition 2.1.5,  $d_r^l$  is properly embedded in  $F^l$ . Q.E.D.

**Lemma 4.1.5.** Suppose that  $j_0 = m^{l+1} \neq \emptyset$ . Then there is a regular neighborhood of  $\partial W_{j_0}^l$  in  $\partial_+ \mathcal{V}_-$ , say  $\partial W_{j_0}^l \times I$ , satisfying the following conditions:

(1)  $\partial W_{j_0}^l \times I \cap F^l = (d_{j_0}^l \times I)_{j_0} \cup_{r \in I(w_{j_0}, l)} (d_r^l \times I)_{j_0}$ ,  $(\partial d_{j_0}^l) \times I, (\partial d_r^l) \times I \subset \cup_f \partial E_f^l - \cup_{\gamma \in m(l)} e_\gamma^l \times I$ .

(2) If  $j_0 < \gamma$ , then  $(d_{j_0}^l \times I)_{j_0}$  is disjoint from  $e_\gamma^l \times I$ , if  $r \in I(w_{j_0}, l)$  and  $r < \gamma$ , then  $(d_r^l \times I)_{j_0}$  is disjoint from  $e_\gamma^l \times I$ .

(3) If  $j_0 > \gamma$ , then each component of  $(d_{j_0}^l \times I)_{j_0} \cap e_\gamma^l \times I$  is  $(c \times I)_{j_0} \subset e_\gamma^l \times (0, 1)$  where  $c \subset \text{int}d_{j_0}^l$  is a core of  $e_\gamma^l \times I$ , if  $r \in I(w_{j_0}, l)$  and  $r > \gamma$ , then each component of  $(d_r^l \times I)_{j_0} \cap (e_\gamma^l \times I)$  is  $(c \times I)_{j_0} \subset e_\gamma^l \times (0, 1)$  where  $c \subset \text{int}d_r^l$  is a core of  $e_\gamma^l \times I$ .

(4) If  $j \neq j_0$  and  $j \notin I(w_{j_0}, l)$ , then  $d_j^l$  is disjoint from  $\partial W_{j_0}^l \times I$ .

(5) For each  $i \geq l+1$ ,  $\partial c_i^l$  is disjoint from  $\partial W_{j_0}^l \times I$ .

(6) If  $j_0 > \gamma$ , then  $(\text{int}d_{i,0} \times I)_{j_0}$  is disjoint from  $e_\gamma^l \times I$  for  $\delta(0) \leq i \leq \theta(0)$ .

(7) For each  $i \geq l+2$  with  $s(v_i) = -$ ,  $\partial W_{j_0}^l \times I$  is disjoint from  $\partial V_i^l - c_i^l \cup_{r \in I(v_i, l)} v_r^l$ , for each  $j \notin m(l+1)$  with  $s(w_j) = -$ ,  $\partial W_{j_0}^l \times I$  is disjoint from  $\partial W_j^l$ .

(8)  $\partial W_{j_0}^l \times I \cap D_{0,\alpha}^* \subset a_\alpha^0 \times I$ .

**Proof.** Now  $F^l = \cup_f E_f^l \cup_{\gamma \in m(l)} e_\gamma^l \times I$ . By Lemma 3.1.1 and Lemma 4.1.4(1),  $d_j^l, d_r^l$  are regular in  $F^l$ . By Definition 2.1.5 and Lemma 2.1.6,  $\partial d_j^l, \partial d_r^l$  are disjoint from  $e_\gamma^l \times I$  for each  $\gamma \in m(l)$ . Hence (1) follows from Proposition 6(1). (2) follows from Proposition 4(3) and Lemma 4.1.4(1). (3) follows from Proposition 4(2) and Definition 2.1.5. By Proposition 4,  $\{d_j^l \mid j \notin \{1, \dots, n\} - m(l)\}$  is a set of pairwise disjoint arcs in  $F^l$ . Since  $j \neq j_0$  and  $j \notin I(w_{j_0}, l)$ ,  $d_j^l \cap (d_{j_0}^l \cup_{r \in I(w_{j_0}, l)} d_r^l) = \emptyset$ . By (1), (4) holds. Since  $c_i^l \subset F^l$ , by Proposition 4(5),  $\partial c_i^l$  is disjoint from  $\partial W_{j_0}^l$  for each  $i \geq l+1$ . Hence (5) holds. (6) follows from Lemma 3.1.2 and Lemma 3.3.1. (7) follows from Proposition 6. (8) follows from Lemma 3.5.7. Q.E.D.

## 4.2 The proofs of Propositions 4-6 for one special case

In this section, we shall prove Propositions 4-6 for the special case:  $k = l+1$  and  $L(c_{l+1}^l) = \emptyset$ .

**The proof of Proposition 4.** Suppose that  $L(c_{l+1}^l) = \emptyset$ . Now by Corollary 3.2.5, for each  $j \notin m(l)$  and  $\gamma \in m(l)$ ,  $d_j^l$  and  $e_\gamma^l \times I$  are disjoint from  $c_{l+1}^l \times I$  where  $c_{l+1}^l \times I$  is a regular neighborhood of  $c_{l+1}^l$  in  $E^l$ . Obviously,  $c_i^l$  is disjoint from  $c_{l+1}^l \times I$  for  $i \geq l+2$ . Now let  $F^{l+1} = F^l - c_{l+1}^l \times (-1, 1)$ . We denote by  $E_f^{l+1}$  the disk  $E_f^l$  for  $1 \leq f \leq l$ ,  $E_0^{l+1}, E_{l+1}^{l+1}$  the two components of  $E_0^l - c_{l+1}^l \times (-1, 1)$ . Let  $c_i^{l+1} = c_i^l$ ,  $d_j^{l+1} = d_j^l$  for  $j \notin m(l)$ ,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$  for  $\gamma \in m(l)$ . Specially, let  $m^{l+1} = \emptyset$ ,  $e_{m^{l+1}}^{l+1} = \emptyset$ . Now if  $c_i^l \subset E_0^l$ , then  $c_i^{l+1}$  lies in one of  $E_0^{l+1}$  and  $E_{l+1}^{l+1}$ . Similarly, if  $d_{j,0}^l \neq \emptyset$ , then  $d_{j,0}^l$  lies in one of  $E_0^{l+1}$  and  $E_{l+1}^{l+1}$ . Since  $m^{l+1} = \emptyset$ ,  $I(w_\gamma, l+1) = I(w_\gamma, l)$ . Hence Proposition 4 holds. Q.E.D.

**The proof of Proposition 5.** By the above argument,  $d_j^{l+1} - \cup_{\gamma < j} \text{inte}_\gamma^{l+1} \times I = d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I$ ,  $c_i^{l+1} = c_i^l$  for  $i \geq l+2$ . Note that  $j \notin L(c_{l+1}^l)$ . Hence Proposition 5 holds. Q.E.D.

**The proof of Proposition 6.** Since  $m^{l+1} = \emptyset$ ,  $m(l+1) = m(l)$ . By Definition 3.2.1,  $I(w_j, l+1) = I(w_j, l)$ . Now we denote by  $W_j^{l+1}$  the disk  $W_j^l$  for  $j \notin m(l)$  with  $s(w_j) = -$ . Then  $W_j^{l+1} \cap F^{l+1} = d_j^{l+1} \cup_{r \in I(w_j, l+1)} d_r^{l+1}$ .

By Definition 2.3.1,  $I(v_i, l+1) = I(v_i, l) - \{l+1\}$ . We denote by  $V_i^{l+1}$  the disk  $V_i^l$  for

$i \geq l + 2$  with  $s(v_i) = -$ . Since  $c_{i+1}^l$  is disjoint from  $F^{l+1}$  and  $d_j^{l+1}$ . Hence  $V_i^{l+1} \cap F^{l+1} = c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_r^{l+1}$ , and  $V_i^{l+1} \cap W_j^{l+1} = (c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_r^{l+1}) \cap (d_j^{l+1} \cup_{r \in I(w_j, l+1)} d_r^{l+1}) = V_i^{l+1} \cap W_j^{l+1} \cap F^{l+1}$ . Q.E.D.

### 4.3 Constructions

By the argument in Section 4.2, in the following argument, we shall assume that  $m^{l+1} \neq \emptyset$ . In this section, we shall construct  $c_i^{l+1}$  for  $i \geq l+2$ ,  $d_j^{l+1}$  for  $j \notin m(l+1) = m(l) \cup \{m^{l+1}\}$ ,  $e_\gamma^{l+1} \times I$  for  $\gamma \in m(l+1)$ , and  $F^{l+1}$  from  $c_i^l, d_j^l, e_\gamma^l \times I, F^l$ .

Let  $\partial W_{j_0}^l \times I$  be a regular neighborhood of  $\partial W_{j_0}^l$  in  $\partial_+ \mathcal{V}_-$  in Lemma 4.1.5. Let  $c_{l+1}^l \times I$  be a regular neighborhood of  $c_{l+1}^l$  in  $E_0^l$ . By Lemma 4.1.3,  $\partial W_{j_0}^l$  intersects  $c_{l+1}^l \times I$  in an arc  $a \subset \text{int} d_{0,0}$ , and  $\partial W_{j_0}^l \times I$  intersects  $c_{l+1}^l \times I$  in  $a \times I$ . (See Figure 13.)

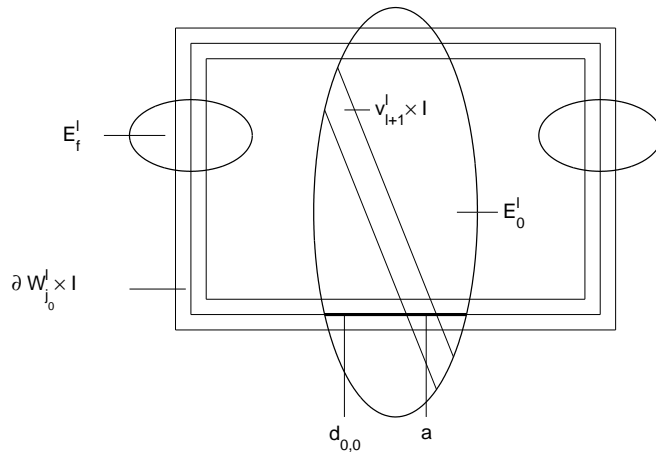


Figure 13

**Definition 4.3.1.** Let  $F^{l+1} = (F^l - c_{l+1}^l \times (-1, 1)) \cup (\partial W_{j_0}^l - \text{inta}) \times I$ .

**Lemma 4.3.2.**  $F^{l+1}$  is a compact surface in  $\partial_+ \mathcal{V}_-$  such that  $c_{j+1}^l$  is disjoint from  $F^{l+1}$ .

**Proof.** By Lemma 4.1.4,  $d_{j_0}^l$  and  $d_r^l$  for  $r \in I(w_{j_0}, l)$  are properly embedded in  $F^l$ . By Lemma 4.1.5(1) and Lemma 4.1.3, the lemma holds. Q.E.D.

**Definition 4.3.3.** Let  $E_f^{l+1} = E_f^l$  for  $1 \leq f \leq l$ . Let  $E_0^{l+1}$  be the component of  $E_0^l - c_{l+1}^l \times (-1, 1)$  containing  $c_{l+1}^l \times \{-1\}$ ,  $E_{l+1}^{l+1}$  be the component of  $E_0^l - c_{l+1}^l \times (-1, 1)$  containing  $c_{l+1}^l \times \{1\}$ .

**Definition 4.3.4.** Let  $c_i^{l+1} = c_i^l$  for  $i \geq l+2$ .

**Construction (\*)**

By Lemma 3.3.3, in  $F^l$ ,  $(\cup_{j \notin m(l)} d_j^l \cup_{\gamma \in m(l)} e_\gamma^l \times I) \cap c_{l+1}^l \times I = (\cup_\alpha (d_{0,\alpha} \times I)_{\gamma_\alpha}) \cap c_{l+1}^l \times I$  where  $(d_{0,\alpha} \times I)_{\gamma_\alpha}$  is defined in Definition 3.2.1. By Lemma 4.1.4,  $(d_{j_0}^l \times I)_{\gamma_0} = d_{j_0}^l$ . Note that  $j_0 = m^{l+1}$ . Hence,  $(\cup_{j \notin m(l+1)} d_j^l \cup_{\gamma \in m(l)} e_\gamma^l \times I) \cap c_{l+1}^l \times I = (\cup_{\alpha \neq 0} (d_{0,\alpha} \times I)_{\gamma_\alpha}) \cap c_{l+1}^l \times I$ . By Definition 3.5.1,  $d_{0,\alpha} \subset a_\alpha$ .

Suppose that  $\alpha \neq 0$ . If  $j_\alpha \in I(w_{\gamma_\alpha})$  for  $\gamma_\alpha \in m(l)$ , then  $\gamma_\alpha > j_\alpha > j_0$ . If  $j_\alpha \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ , then  $(d_{j_\alpha}^l \times I)_{\gamma_\alpha} = d_{j_\alpha}^l$ . By Lemma 4.1.5(2) and (4),  $\partial W_{j_0}^l \times I$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha} \subset (d_{j_\alpha}^l \times I)_{\gamma_\alpha}$  for  $\alpha \neq 0$ . By Lemma 4.1.5(8),  $\partial W_{j_0}^l \times I \cap D_{0,\alpha}^* \subset a_\alpha^0 \times I$ . Now  $a_1^0 \cup a_{-1}^0$  separates  $\cup_{\alpha \neq 0} D_{0,\alpha}^*$  into two disks  $\cup_{\alpha < 0} D_{0,\alpha}^*$  and  $\cup_{\alpha > 0} D_{0,\alpha}^*$ . We may assume that

(0)  $D_{0,\alpha}^* \cap a_\alpha^0 \times I \subset a_\alpha^0 \times [0, 1]$  for  $\alpha > 0$ , and  $D_{0,\alpha}^* \cap a_\alpha^0 \times I \subset a_\alpha^0 \times [-1, 0]$  for  $\alpha < 0$ .

Now  $\partial W_{j_0}^l \times I \cup_\alpha D_{0,\alpha}^*$  is as in Figure 14.

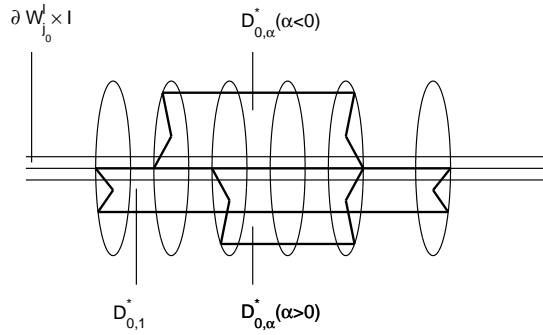


Figure 14

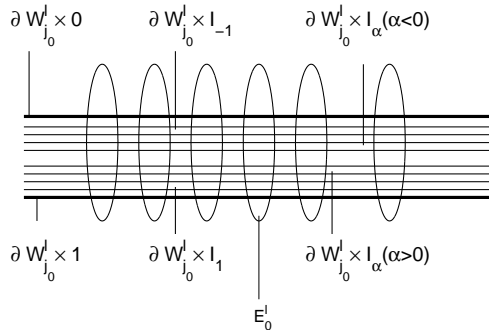


Figure 15

Now let  $I_\alpha = [t_1^\alpha, t_2^\alpha]$  be a sub-interval of  $[0, 1]$  such that

- (1) If  $0 < \alpha < \beta$ , then  $t_2^\alpha > t_1^\alpha > t_2^\beta > t_1^\beta > 1/2$ .
- (2) If  $\alpha < \beta < 0$ , then  $1/2 > t_2^\alpha > t_1^\alpha > t_2^\beta > t_1^\beta$ .

Now  $\partial W_{j_0}^l \times I_\alpha$  is as in Figure 15.

Let  $t_\alpha$  be the center point of  $I_\alpha$ , and  $a_{0,\alpha}^*$  be an arc connecting  $\partial_1 a_\alpha$  to  $\partial_1 a_\alpha^0 \times t_\alpha$  in  $E_{f_{\delta(0,\alpha)},\alpha}^l$  such that

- (3)  $a_{0,\alpha} - \partial W_{j_0}^l \times I = a_{0,\alpha}^* - \partial W_{j_0}^l \times I$ , where  $a_{0,\alpha}$  is as in Lemma 3.5.5 and  $a_{0,\alpha}^* \cap \partial W_{j_0}^l \times I$  is an arc in  $d_{\delta(0,\alpha),0} \times I \subset \partial W_{j_0}^l \times I$  as in Figure 16. By Definition 3.5.1,  $d_{\delta(0,\alpha),0} \subset a_\alpha^0$ .

Similarly,  $b_{0,\alpha}^*$  be an arc connecting  $\partial_2 a_\alpha$  to  $\partial_2 a_\alpha^0 \times t_\alpha$  in  $E_{f_{\theta(0,\alpha)},\alpha}^l$  such that

- (4)  $b_{0,\alpha} - \partial W_{j_0}^l \times I = b_{0,\alpha}^* - \partial W_{j_0}^l \times I$  and  $b_{0,\alpha}^* \cap \partial W_{j_0}^l \times I$  is an arc in  $d_{\theta(0,\alpha),0} \times I \subset \partial W_{j_0}^l \times I$ .

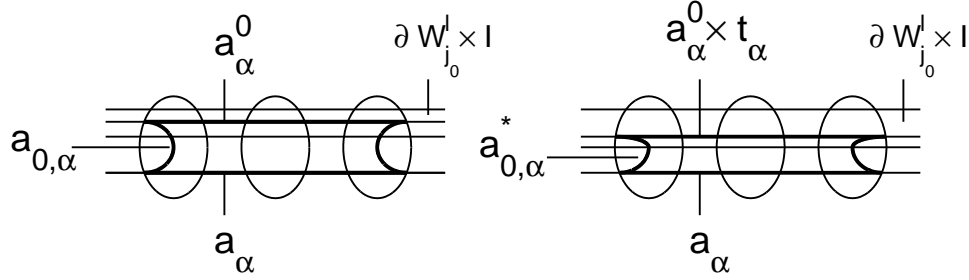


Figure 16

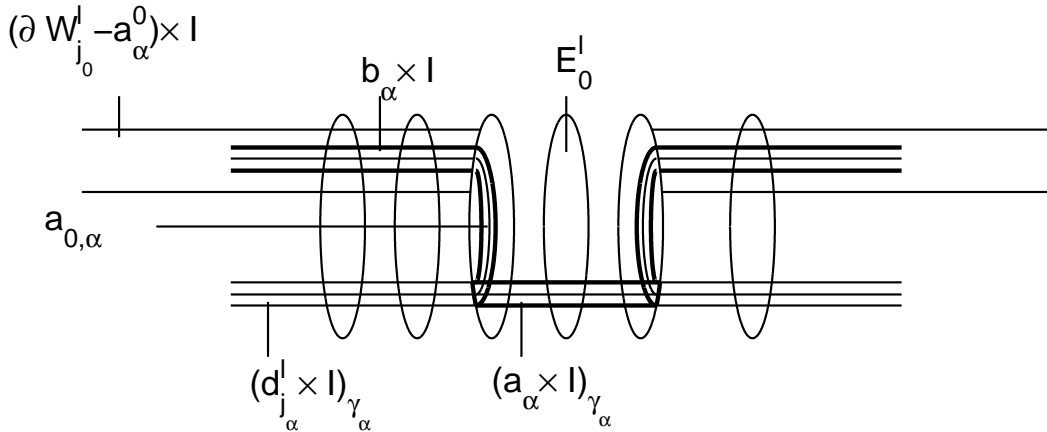


Figure 17



By Definition 3.5.1,  $d_{\theta(0,\alpha),0} \subset a_\alpha^0$ . For  $\alpha \neq 0$ , let  $b_\alpha = (\partial W_{j_0} - a_\alpha^0) \times \{t_\alpha\} \cup a_{0,\alpha}^* \cup b_{0,\alpha}^*$  as in Figure 17. Then  $b_\alpha$  is an arc in  $F^{l+1}$  by Lemma 3.5.8. Now let  $b_\alpha \times I$  be a neighborhood of  $b_\alpha$  in  $F^{l+1}$  satisfying the following conditions:

$$(5) \quad b_\alpha \times I \cap \partial W_{j_0}^l \times I = (\partial W_{j_0} - \text{inta}_\alpha^0) \times I_\alpha.$$

(6) If  $s(w_{\gamma_\alpha}) = -$ , then  $(\partial b_\alpha) \times I = ((\partial a_\alpha) \times I)_{\gamma_\alpha}$ ; if  $s(w_{\gamma_\alpha}) = +$  or  $\gamma_\alpha = \emptyset$ , then  $\partial b_\alpha = ((\partial a_\alpha) \times I)_{\gamma_\alpha}$ .

If  $j_\alpha \in I(w_{\gamma_\alpha}, l)$  with  $s(w_{\gamma_\alpha}) = -$ , then there is a homeomorphism  $H_\alpha$  from  $(a_\alpha \times I)_{\gamma_\alpha}$  to  $b_\alpha \times I$  such that  $H_\alpha$  is an identifying map on  $(\partial b_\alpha) \times I = (\partial a_\alpha) \times I$ . If  $s(w_{\gamma_\alpha}) = +$  or  $\gamma_\alpha = \emptyset$ , then there is a homeomorphism  $H_\alpha$  from  $(a_\alpha \times I)_{\gamma_\alpha}$  to  $b_\alpha = b_\alpha \times \{0\}$ . In this case,  $(a_\alpha \times I)_{\gamma_\alpha} = a_\alpha$ . (See Definition 3.2.1.) Q.E.D.(Construction(\*))

**Lemma 4.3.5.** (1) If  $\alpha \neq \beta$ , then  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ .

(2)  $H = \cup H_\alpha$  is an injective map from  $\cup_\alpha (a_\alpha \times I)_{\gamma_\alpha}$  to  $\cup_\alpha b_\alpha \times I$ .

(3) If  $j \notin m(l+1)$ , and  $d_j^l \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I) \neq \emptyset$ , then  $d_j^l \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I) \subset \cup_{\beta \neq 0} (a_\beta \times I)_{\gamma_\beta}$ . Furthermore, either  $j = j_\lambda$  or  $j > \gamma_\lambda$  for some  $\lambda$ .

(4)  $e_\gamma^l \times I \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I) \subset \cup_{\beta \neq 0} (a_\beta \times I)_{\gamma_\beta}$ . If  $e_\gamma^l \times I \cap (\text{inta}_{0,\alpha}^* \times I \cup \text{intb}_{0,\alpha}^* \times I) \neq \emptyset$ , then  $e_\gamma^l \times I \cap (\text{inta}_{0,\alpha}^* \times I \cup \text{intb}_{0,\alpha}^* \times I) \subset \cup_{\beta \neq 0} (a_\beta \times I)_{\gamma_\beta}$ , and  $\gamma \geq \gamma_\lambda$  for some  $\lambda$ .

(5)  $b_\alpha \times I \subset F^{l+1}$ .

**Proof.** (1) By Construction(\*) (5) and (6), we only need to prove that  $(a_{0,\alpha}^* \cup b_{0,\alpha}^*) \cap (a_{0,\beta}^* \cup b_{0,\beta}^*) = \emptyset$ . Without loss of generality, we assume that  $0 < \alpha < \beta$ . By Lemma 3.4.7,  $\delta(0, \alpha) \leq \delta(0, \beta) \leq \theta(0, \beta) \leq \theta(0, \alpha)$ .

Suppose that  $\delta(0, \alpha) < \delta(0, \beta) \leq \theta(0, \beta)$ . Since  $f_{i,\alpha} = f_{i,0}$  for  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$   $f_{i,\beta} = f_{i,0}$  for  $\delta(0, \beta) \leq i \leq \theta(0, \beta)$ . By Lemma 3.3.4,  $f_{\delta(0,\alpha),\alpha} \neq f_{\delta(0,\beta),\beta}, f_{\theta(0,\beta),\beta}$ .  $a_{0,\alpha}^* \cap (a_{0,\beta}^* \cup b_{0,\beta}^*) = \emptyset$ .

Suppose that  $\delta(0, \alpha) = \delta(0, \beta) \leq \theta(0, \beta)$ . Since  $a_\alpha$  separates  $a_\alpha^0 = a_\beta^0$  and  $a_\beta$  in  $D_{0,\beta}^*$ . Now by the choice of  $I_\alpha$ , (1) holds.

(2) By Lemma 3.5.4,  $(a_\alpha \times I)_{\gamma_\alpha} \cap (a_\beta \times I)_{\gamma_\beta} = \emptyset$  for  $\alpha \neq \beta$ . By (1),  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ . Hence (2) holds.

(3, 4) By Construction(\*) (3) and (4),  $a_{0,\alpha} - \partial W_{j_0}^l \times I = a_{0,\alpha}^* - \partial W_{j_0}^l \times I$  and  $a_{0,\alpha}^* \cap \partial W_{j_0}^l \times I$

is an arc in  $d_{\delta(0,\alpha)} \times I$ . By Lemma 4.1.5(2), (4) and (6),  $d_j^l$  and  $e_\gamma^l \times I$  are disjoint from  $intd_{i,0} \times I$  for  $j \neq j_0 = m^{l+1}$  and  $\gamma \in m(l)$ . Now by Lemma 3.5.5, (3) and (4) holds.

(5) Since  $d_{0,0} \subset a_\alpha^0$ ,  $(\partial W_{j_0}^l - inta_\alpha^0) \times I_\alpha \subset F^{l+1}$ . By Lemma 3.5.8,  $a_{0,\alpha}^* \cup b_{0,\alpha}^*$  is disjoint from  $c_{l+1}^l \times I$ . Hence (5) holds. Q.E.D.

**Definition 4.3.6.** Let  $d_j^{l+1} = (d_j^l - \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}) \cup H(d_j^l \cap (\cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}))$  for  $j \notin m(l+1)$ .

**Lemma 4.3.7.**  $\{c_i^{l+1} \mid i \geq l+2\}$  is a set of pairwise disjoint arcs properly embedded in  $F^{l+1}$  which lies in one of  $E_f^{l+1}$ .

**Proof.** Since  $i \geq l+2$ ,  $c_i^{l+1} = c_i^l$  is disjoint from  $c_{l+1}^l \times I$ . Hence  $c_i^{l+1}$  lies in  $E_f^{l+1}$  for some  $f$  by Proposition 4(4). Furthermore, if  $c_i^l \subset E_0^l$ , then  $c_i^{l+1}$  lies in one of  $E_0^{l+1}$  and  $E_{l+1}^{l+1}$ . By Proposition 4(4) and Lemma 4.1.5(5),  $c_i^{l+1}$  is properly embedded in  $F^{l+1}$ . Q.E.D.

**Lemma 4.3.8.** (1)  $\{d_j^{l+1} \mid j \notin m(l+1)\}$  is a set of pairwise disjoint arcs in  $F^{l+1}$  such that  $\partial d_j^{l+1} = \partial d_j^l$ .

(2) If  $j \in \{1, \dots, n\} - m(l+1)$  and  $j \neq j_\alpha$  for each  $\alpha$ , then  $d_j^{l+1} = (d_j^l - \cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha < j} H_\alpha(d_j^l \cap (a_\alpha \times I)_{\gamma_\alpha})$ .

(3) If  $j = j_\beta$  for some  $\beta \neq 0$ , then  $d_j^{l+1} = (d_j^l - a_\beta \cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha < j} H_\alpha(d_j^l \cap (a_\alpha \times I)_{\gamma_\alpha}) \cup b_\beta$ .

(4) If  $j < j_0$ , then  $d_j^l = d_j^{l+1}$ .

**Proof.** (1) By Proposition 4(2) and (3), each component  $d_j^l \cap e_{\gamma_\alpha}^l \times I$  is a core of  $e_{\gamma_\alpha}^l \times (0, 1)$  if  $j \notin I(w_{\gamma_\alpha}, l)$ . Hence each component of  $d_j^l \cap (a_\alpha \times I)_{\gamma_\alpha}$  is a core of  $(a_\alpha \times I)_{\gamma_\alpha}$  even if  $j = j_\alpha$ . By Lemma 4.3.5,  $d_j^l \cap (\cup_{\alpha \neq 0} a_{0,\alpha}^* \cup b_{0,\alpha}^*) \subset \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}$  and  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ . By Lemma 3.3.3 and Lemma 4.1.4,  $(\cup_{j \notin m(l+1)} d_j^l) \cap c_{l+1}^l \times I = (\cup_{\alpha \neq 0} (d_{0,\alpha} \times I)_{\gamma_\alpha}) \cap c_{l+1}^l \times I$ . By Definition 3.5.1,  $d_{0,\alpha} \subset a_\alpha$ . By Lemma 4.3.5(5),  $d_j^{l+1}$  is an arc in  $F^{l+1}$ . By Proposition 4,  $d_j^l \cap d_r^l = \emptyset$  for  $j \neq r$ . Since  $H$  is injective,  $d_j^{l+1} \cap d_r^{l+1} = \emptyset$ .

(2) If  $j \neq j_\alpha$  for each  $\alpha \neq 0$  and  $j < \gamma_\alpha$ , then, by Proposition 4(3),  $d_j^l$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha} \subset e_{\gamma_\alpha}^l \times I$ . Hence (2) holds.

(3) Suppose that  $j = j_\beta$  for some  $\beta \neq 0$ . Then  $a_\beta = a_\beta \times \{0\} \subset d_j^l$ . Furthermore,  $H_\beta(a_\beta) = b_\beta$ . By the argument in (2), (3) holds.

(4) If  $j < j_0$ , then  $j < j_\alpha$  for each  $\alpha \neq 0$ . By Proposition 4(3),  $d_j^l \cap (a_\alpha \times I)_{\gamma_\alpha} = \emptyset$  for each  $\alpha$ . By (2), (4) holds. Q.E.D.

In fact, if  $j > j_0$ , then  $d_j^{l+1}$  is obtained by doing band sums with copies of  $\partial W_{j_0}^l$  to  $d_j^l$ . See Lemma 4.6.3.

**Definition 4.3.9.** (1) Let  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$  for  $\gamma \in m(l)$  with  $s(w_\gamma) = +$ .

(2) Suppose that  $r \in I(w_\gamma, l)$  for some  $\gamma \in m(l)$  with  $s(w_\gamma) = -$ . If  $\gamma > j_0$ , then let  $(d_r^{l+1} \times I)_\gamma = ((d_r^l \times I)_\gamma - \cup_\alpha (a_\alpha \times I)_{\gamma_\alpha}) \cup H((d_r^l \times I)_\gamma \cap (\cup_\alpha (a_\alpha \times I)_{\gamma_\alpha}))$ . If  $\gamma \leq j_0$ , then let  $(d_r^{l+1} \times I)_\gamma = (d_r^l \times I)_\gamma$ .

(3) Suppose that  $\gamma \in m(l)$  with  $s(w_\gamma) = -$ . Let  $e_\gamma^{l+1} = (e_\gamma^l - \cup_{r \in I(w_\gamma, l)} d_r^l) \cup_{r \in I(w_\gamma, l)} d_r^{l+1}$ ,  $e_\gamma^{l+1} \times I = (e_\gamma^l \times I - \cup_{r \in I(w_\gamma, l)} (d_r^l \times I)_\gamma) \cup_{r \in I(w_\gamma, l+1)} (d_r^{l+1} \times I)_\gamma$ . Specially, let  $e_{m^{l+1}}^{l+1} = \partial W_{j_0}^l - \text{int} d_{j_0}^l$ ,  $e_{m^{l+1}}^{l+1} \times I = (\partial W_{j_0}^l - \text{int} d_{j_0}^l) \times I$ .

**Lemma 4.3.10.** (1) For each  $\gamma \in m(l+1)$ ,  $e_\gamma^{l+1} \times I$  is a disk in  $F^{l+1}$  such that  $(\partial e_\gamma^l) \times I = (\partial e_\gamma^{l+1}) \times I$  for  $\gamma \in m(l)$  and  $(\partial e_{m^{l+1}}^{l+1}) \times I = (\partial d_{j_0}^l) \times I$ .

(2) Suppose that  $r \in I(w_\gamma, l)$  for some  $j_0 < \gamma \in m(l)$  with  $s(w_\gamma) = -$ . If  $r \neq j_\alpha$  for each  $\alpha$ , then  $(d_r^{l+1} \times I)_\gamma = ((d_r^l \times I)_\gamma - \cup_{\gamma_\alpha < r} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha < r} H_\alpha((d_r^l \times I)_\gamma \cap (a_\alpha \times I)_{\gamma_\alpha})$ .

(3) Suppose that  $s(w_{\gamma_\beta}) = -$ . Then  $(d_{j_\beta}^{l+1} \times I)_{\gamma_\beta} = ((d_{j_\beta}^l \times I)_{\gamma_\beta} - (a_\beta \times I)_{\gamma_\beta} \cup_{\gamma_\alpha < j_\beta} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha < j_\beta} H_\alpha((d_{j_\beta}^l \times I)_{\gamma_\beta} \cap (a_\alpha \times I)_{\gamma_\alpha}) \cup b_\beta \times I$ .

**Proof.** Suppose that  $s(w_\gamma) = +$ . By Proposition 4(1) and (5),  $e_\gamma^l \times I$  is disjoint from  $c_{l+1}^l \times I \subset F^l$ . Hence  $e_\gamma^{l+1} \times I = e_\gamma^l \times I \subset F^{l+1}$  is a disk.

Suppose that  $s(w_\gamma) = -$ . There are four cases:

Case 1.  $\gamma < j_0$ .

Now if  $r \in I(w_\gamma)$ , then  $r < \gamma < j_0$ . By Lemma 4.1.2 and Lemma 4.3.7,  $(d_r^{l+1} \times I)_\gamma = (d_r^l \times I)_\gamma \subset e_\gamma^l \times I$  is disjoint from  $c_{l+1}^l \times I$ . By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I \subset F^{l+1}$  is a disk.

Case 2. By the construction,  $e_{m^{l+1}}^{l+1} \times I$  is a disk in  $F^{l+1}$ .

Case 3.  $\gamma > j_0$  and  $\gamma \neq \gamma_\alpha$  for each  $\alpha$ .

Now  $\gamma \geq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . Since  $\delta(0, \alpha) \leq i \leq \theta(0, \alpha)$ ,  $\gamma_{i,\alpha} = \gamma_{i,0}$ . By Lemma 3.5.3, if  $e_\gamma^l \times I \cap (\text{int} a_\alpha \times I)_{\gamma_\alpha} \neq \emptyset$ , then,  $\gamma > \gamma_\alpha$ . By Proposition 4(1),  $e_\gamma^l \times I \cap e_{\gamma_\alpha}^l \times I =$

$(\cup_{r \in I(w_\gamma, l)} (d_r^l \times I)_\gamma) \cap e_{\gamma_\alpha}^l \times I$ . By Definition 2.1.4, each component of  $e_\gamma^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha}$  is  $(c \times I)_\gamma \subset (a_\alpha \times (0, 1))_{\gamma_\alpha}$  where  $c$  is a core of  $(a_\alpha \times I)_{\gamma_\alpha}$ . By Lemma 3.2.6, if  $r \in I(w_\gamma, l)$  and  $r < \gamma_\alpha$ , then  $(d_r^l \times I)_\gamma$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$ . Hence (2) holds. By Lemma 4.1.5(2),  $e_\gamma^l \times I$  is disjoint from  $W_{j_0}^l \times I_\alpha \subset W_{j_0}^l \times I$ . By Lemma 4.3.5, if  $e_\gamma^l \times I \cap ((inta_{0,\alpha}^* \cup intb_{0,\alpha}^*) \times I)_{\gamma_\alpha} \neq \emptyset$ , then  $e_\gamma^l \times I \cap ((inta_{0,\alpha}^* \cup intb_{0,\alpha}^*) \times I)_{\gamma_\alpha} \subset \cup_{\beta \neq 0} ((inta_\beta \cup intb_\beta) \times I)_{\gamma_\beta}$ . Since  $H$  is injective,  $e_\gamma^{l+1} \times I$  is a disk in  $F^{l+1}$ .

Case 4.  $\gamma = \gamma_\beta$  for some  $\beta$ .

Now  $(a_\beta \times I)_{\gamma_\beta} \subset (d_{j_\beta}^l \times I)_{\gamma_\beta} \subset e_{\gamma_\beta}^l \times I$ . By Definition 4.3.9 and the argument in Case 3,  $e_\gamma^{l+1} \times I$  is a disk in  $F^{l+1}$  and (3) holds. Q.E.D.

#### 4.4 $F^{l+1}$ is a surface generated by an abstract tree in $\partial+V_-$

By the construction,  $F^{l+1}$  is a surface in  $\partial+V_-$ . In this section, we shall prove that  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^{l+1}$  is an abstract tree and  $F^{l+1}$  is a surface generated by  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^{l+1}$ , which satisfies Proposition 4(1).

**Lemma 4.4.1.** (1)  $F^{l+1} = \cup_{f=0}^{l+1} E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^{l+1} \times I$ .

(2) For  $\gamma \neq \lambda \in m(l+1)$ ,  $(\partial e_\gamma^{l+1}) \times I \cap (\partial e_\lambda^{l+1}) \times I = \emptyset$ .

**Proof.** By the construction,  $F^{l+1} = (F^l - c \times (-1, 1)) \cup (\partial W_{j_0}^l - inta) \times I$ , where  $a \subset d_{0,0}$ . Since  $d_{j_0}^l \subset F^l$ ,  $F^{l+1} = (F^l - c \times (-1, 1)) \cup e_{m^{l+1}}^{l+1} \times I$ . Hence (1) follows from the construction.

Since  $F^l$  is generated by  $\cup_f E_f^l \cup_{\gamma \in m(l)} e_\gamma^l$ . Hence, by Lemma 4.3.10,  $(\partial e_\gamma^{l+1}) \times I \cap (\partial e_\lambda^{l+1}) \times I = \emptyset$  for  $\gamma \neq \lambda \in m(l)$ . By Lemma 4.3.10 and Lemma 4.1.5(1),  $(\partial e_\gamma^{l+1}) \times I \cap (\partial e_{m^{l+1}}^{l+1}) \times I = \emptyset$  for  $\gamma \in m(l)$ . Q.E.D.

**Lemma 4.4.2.** If  $\lambda \in m(l+1)$  with  $s(w_\lambda) = +$ , then  $inte_\gamma^{l+1} \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I$ .

**Proof.** Since  $s(w_\lambda) = +$ , by Lemma 4.1.3,  $\lambda \neq j_0 = m^{l+1}$ . By Proposition 4(1) and (5),  $e_\gamma^l \times I$  is disjoint from  $c_{l+1}^l \times I \subset E_0^l$ . By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$ .

Suppose that  $\lambda < j_0$ . By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$  for  $\gamma < \lambda$ . Note that  $E_0^{l+1}, E_{l+1}^{l+1} \subset E_0^l$  and  $E_f^{l+1} = E_f^l$  for  $1 \leq f \leq l$ . In this case,  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I =$

$\cup_f E_f^l \cup_{\gamma < \lambda} e_\gamma^l \times I - c_{l+1}^l \times (-1, 1)$ . Hence  $inte_\lambda^{l+1} \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I$ .

Suppose that  $\lambda > j_0$ . By Proposition 4(1),  $inte_\lambda^l \times I$  is disjoint from  $\cup_f E_f^l \cup_{\gamma < \lambda} e_\gamma^l \times I$ . By Lemma 4.1.5(1) and (2),  $e_\lambda^l \times I$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I \subset \partial W_{j_0}^l \times I$ . By Definition 4.3.9,  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I = (\cup_f E_f^l \cup_{\gamma < \lambda} e_\gamma^l \times I - c_{l+1}^l \times (-1, 1)) \cup e_{m^{l+1}}^{l+1} \times I$ . Hence  $inte_\lambda^{l+1} \times I = inte_\lambda^l \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I$ . Q.E.D.

**Lemma 4.4.3.** Suppose that  $\lambda \in m(l+1)$  with  $s(w_\lambda) = -$ . Then  $inte_\lambda^{l+1} \times I \cap (\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I) = \cup_{r \in I(w_\lambda, l+1)} d_r^{l+1} \times I$ .

**Proof.** By assumption, Proposition 4 holds for  $k = l$ . Hence  $inte_\lambda^l \times I \cap (\cup_f E_f^l \cup_{\gamma < \lambda} e_\gamma^l \times I) = \cup_{r \in I(w_\lambda, l)} d_r^l \times I$  for  $\lambda \in m(l)$  with  $s(w_\lambda) = -$ .

Since  $s(w_\lambda) = -$ , by Lemma 2.2.5 and Lemma 4.1.3,  $j_0 = m^{l+1} \notin I(w_\lambda, l)$ . By Definition 2.3.1,  $I(w_\lambda, l+1) = I(w_\lambda, l) - \{m^{l+1}\} = I(w_\lambda, l)$ . Now there are four cases:

Case 1.  $\lambda < j_0 = m^{l+1}$ .

In this case,  $\{\gamma \mid \lambda > \gamma \in m(l+1)\} = \{\gamma \mid \lambda > \gamma \in m(l)\}$ .

By Definition 4.3.9,  $e_\lambda^{l+1} \times I = e_\lambda^l \times I$  and  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$  for  $\gamma \leq \lambda$ . By Lemma 4.3.8 and Lemma 2.2.4,  $d_r^{l+1} = d_r^l$  for  $r \in I(w_\lambda, l)$  or  $r \in I(w_\gamma, l)$ . Hence  $inte_\lambda^{l+1} \times I \cap (\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e_\gamma^{l+1} \times I) = inte_\lambda^l \times I \cap (\cup_f E_f^l \cup_{\gamma < \lambda} e_\gamma^l \times I - c_{l+1}^l \times (-1, 1))$ . By Lemma 4.1.2,  $e_\gamma^l \times I$  is disjoint from  $c_{l+1}^l \times I$ . By Definition 4.3.9, the lemma holds.

Case 2.  $\lambda = j_0 = m^{l+1}$ .

By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$  for  $\gamma < \lambda$ . and  $e_{m^{l+1}}^{l+1} \times I = (\partial W_{j_0} - int d_{j_0}^l) \times I$ . If  $r \in I(w_{j_0}, l)$ , then  $r < j_0$ . By Lemma 4.3.8,  $d_r^{l+1} = d_r^l$  is disjoint from  $c_{l+1}^l \times I$ . By Lemma 4.1.5(1) and (2),  $d_r^{l+1} \times I \subset \cup_f E_f^{l+1} \cup_{\gamma < j_0} e_\gamma^{l+1} \times I = \cup_f E_f^l \cup_{\gamma < j_0} e_\gamma^l \times I - c_{l+1}^l \times (-1, 1)$ . Hence the lemma holds.

Case 3.  $\lambda > j_0 = m^{l+1}$  and  $\lambda \neq \gamma_\alpha$  for each  $\alpha$ .

In this case,  $\{\gamma \mid \gamma < \lambda \in m(l+1)\} = \{\gamma \mid \gamma < \lambda \in m(l)\} \cup \{m^{l+1}\}$ .

By Lemma 4.1.5(1) and (2),  $e_\lambda^l \times I$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I \subset \partial W_{j_0}^l \times I$ . By Proposition 4(1),  $inte_\lambda^l \times I - \cup_{r \in I(w_\lambda, l)} d_r^l \times I$  is disjoint from  $\cup_f E_f^l \cup_{\gamma < \lambda \in m(l)} e_\gamma^l \times I$ . By Definition 4.3.9, for each  $\gamma \in m(l)$ ,  $e_\gamma^{l+1} \times I \subset e_\gamma^l \times I \cup \partial W_{j_0}^l \times I$ . Hence  $inte_\lambda^l \times I - \cup_{r \in I(w_\lambda, l)} d_r^l \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda \in m(l+1)} e_\gamma^{l+1} \times I$ . Since  $\lambda > j_0$ , by Lemma 4.3.10(2),  $(d_r^{l+1} \times I)_\lambda =$

$$((d_r^l \times I)_\lambda - \cup_{\gamma_\alpha < r} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha < r} H_\alpha((d_r^l \times I)_\lambda \cap (a_\alpha \times I)_{\gamma_\alpha}) \subset \cup_f E_f^{l+1} \cup_{\gamma < \lambda \in m(l+1)} e_\gamma^{l+1} \times I.$$

Hence the lemma holds.

Case 4.  $\lambda = \gamma_\beta$ .

By Lemma 4.3.5 and Construction(\*),  $b_\beta \times I \subset (\partial W_{j_0}^l - a_\beta^0) \times I \cup E_{f_{\delta(0,\beta),0}}^{l+1} \cup E_{f_{\theta(0,\beta),0}}^{l+1}$ . By Lemma 4.1.5 and Definition 4.3.9,  $(\partial W_{j_0}^l - a_\beta^0) \times I \subset \cup_f E_f^{l+1} \cup_{\gamma < j_0} e_\gamma^{l+1} \times I$ . By the argument in Case 3 and Lemma 4.3.10(3), the lemma holds Q.E.D.

**Lemma 4.4.4.** If  $\lambda > \gamma \in m(l+1)$ , then each component of  $e_\lambda^{l+1} \cap e_\gamma^{l+1} \times I$  is an arc  $c \subset \text{inte}_\lambda^{l+1}$  which is a core of  $e_\gamma^{l+1} \times (0,1)$ , and each component of  $e_\lambda^{l+1} \times I \cap e_\gamma^{l+1} \times I$  is  $(c \times I)_\lambda \subset e_\gamma^{l+1} \times (0,1)$ .

**Proof.** Recalling the assumption that Proposition 4 holds for  $k = l$ . There are five cases:

Case 1.  $\gamma < \lambda < j_0$ .

Now  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$  and  $e_\lambda^{l+1} \times I = e_\lambda^l \times I$ . Since  $F^l$  is a surface generated by the abstract tree  $\cup_f E_f^l \cup_{\gamma \in m(l)} e_\gamma^l$ , the lemma holds. (See Definition 2.1.4.)

Case 2.  $\gamma < \lambda = j_0$ .

Now  $e_\gamma^{l+1} \times I = e_\gamma^l \times I \subset F^l - c_{l+1}^l \times (-1,1)$ . By Lemma 4.4.3,  $e_{m^{l+1}}^{l+1} \times I \cap (\cup_f E_f^{l+1} \cup_{\gamma < m^{l+1}} e_\gamma^{l+1} \times I) = (\partial d_{j_0}^l) \times I \cup_{r \in I(w_{j_0}, l+1)} d_r^{l+1} \times I$ . By Lemma 4.3.8,  $d_r^{l+1} = d_r^l$ . By Lemma 4.1.5(1), (2), (3) and Lemma 4.4.1(2), the lemma holds.

Case 3.  $\gamma < j_0 = m^{l+1} < \lambda$ .

By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I \subset F^l - c_{l+1}^l \times [-1,1]$ . By Lemma 4.1.5, each component of  $\partial W_{j_0}^l \times I \cap e_\gamma^l \times I$  is  $c \times I \subset e_\gamma^l \times (0,1)$  where  $c$  is a core of  $e_\gamma^l \times (0,1)$ . By Lemma 4.1.5(6),  $(\text{int} d_{i,0} \times I)_{j_0}$  is disjoint from  $e_\gamma^l \times I = e_\gamma^l \times I$ . Note that  $a_\alpha^0 = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,0} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,0}$ . Hence each component of  $\partial W_{j_0}^l \times I \cap e_\gamma^l \times I$  is either in  $(\partial W_{j_0}^l - \text{inta}_\alpha^0) \times I$  or in  $\text{inta}_\alpha^0 \times I$ .

Since  $\gamma < j_0 < j_\alpha$ , by Lemma 4.3.5(4),  $\text{inta}_{0,\alpha}^* \times I \cup \text{int} b_{0,\alpha}^* \times I$  is disjoint from  $e_\gamma^l \times I$ . Since  $j_\alpha > j_0$ , by Lemma 3.2.6,  $(\text{int} d_{i,\alpha} \times I)_{\gamma_\alpha}$  is also disjoint from  $e_\gamma^l \times I$ . Hence each component of  $e_{\gamma_\alpha}^l \times I \cap e_\gamma^l \times I$  is either in  $(e_{\gamma_\alpha}^l - \text{inta}_\alpha) \times I$  or in  $(\text{inta}_\alpha \times I)_{\gamma_\alpha}$ .

Since  $\lambda > j_0$ ,  $\lambda > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$  by Lemma 3.3.1. Hence if  $e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha} \neq \emptyset$ , then, by Lemma 3.5.3,  $\lambda \geq \gamma_\alpha$ . By Definition 2.1.4 and Proposition 4(1), each

component of  $e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha}$  is  $(c \times I)_\lambda \subset (a_\alpha \times I)_{\gamma_\alpha}$  where  $c$  is a core of  $(a_\alpha \times I)_{\gamma_\alpha}$ . Hence each component of  $e_\lambda^l \times I \cap e_\gamma^l \times I$  is either in  $(a_\alpha \times I)_{\gamma_\alpha}$  or in  $e_\lambda^l \times I - (inta_\alpha \times I)_{\gamma_\alpha}$ . By Lemma 4.3.10(2), (3) and Lemma 4.4.3,  $e_\lambda^{l+1} \times I \cap e_\gamma^{l+1} \times I = S_1 \cup S_2$  where  $S_1 = (e_\lambda^l \times I - \cup_{\gamma_\alpha \leq \lambda} (inta_\alpha \times I)_{\gamma_\alpha}) \cap e_\gamma^l \times I$  and  $S_2 = (\cup_{\gamma_\alpha \leq \lambda} H_\alpha(e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha})) \cap e_\gamma^l \times I$ . By the construction,  $H_\alpha((c \times I)_\lambda) = (H_\alpha(c) \times I)_\lambda \subset b_\alpha \times I$  where  $H_\alpha(c)$  is a core of  $b_\alpha \times I$ . By the above argument, the lemma holds.

Case 4.  $\gamma = j_0 < \lambda$ .

By Lemma 4.1.5,  $e_\lambda^l \times I$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I = (\partial W_{j_0}^l - int d_{j_0}^l) \times I$ . By Lemma 4.3.10(2), (3) and Lemma 4.4.3,  $e_\lambda^{l+1} \times I \cap e_{m^{l+1}}^{l+1} \times I = (\cup_{\gamma_\alpha \leq \lambda} H_\alpha(e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha})) \cap e_{m^{l+1}}^{l+1} \times I$ . Note that each component of  $e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha}$  is  $(c \times I)_\lambda \subset (a_\alpha \times I)_{\gamma_\alpha}$ . By Definition 4.3.9,  $H_\alpha((c \times I)_\lambda)$  intersects  $b_\alpha \times I$  in  $(H_\alpha(c) \times I)_\lambda$ . Since  $a_\alpha^0 \subset d_{j_0}^l$ , by Definition 4.3.9,  $e_{m^{l+1}}^{l+1} \times I \subset (\partial W_{j_0}^l - int a_\alpha^0) \times I$ . By Lemma 4.1.5(8) and Construction(\*) (3), (4),  $inta_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I$ . By the construction,  $b_\alpha \times I$  intersects  $e_{m^{l+1}}^{l+1} \times I$  in  $e_{m^{l+1}}^{l+1} \times I_\alpha$ . Since  $I_\alpha \subset (0, 1)$ , by Proposition 4 for  $k = l$ , the lemma holds.

Case 5.  $j_0 < \gamma < \lambda$ . By Lemma 4.1.5,  $e_\gamma^l \times I, e_\lambda^l \times I$  are disjoint from  $\partial W_{j_0} \times I$ . By Lemma 4.3.5,  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ . By Definition 4.3.10(2),  $e_\lambda^{l+1} \times I \cap e_\gamma^{l+1} \times I = S_1 \cup S_2$  where  $S_1 = (e_\lambda^l \times I - \cup_{\gamma_\alpha \leq \lambda, \gamma} (e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha})) \cap (e_\gamma^l \times I - \cup_{\gamma_\alpha \leq \lambda, \gamma} (e_\gamma^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha}))$  and  $S_2 = \cup_{\gamma_\alpha \leq \gamma, \lambda} H(e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha}) \cap H(e_\gamma^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha})$ . Since  $H$  is injective, the lemma holds. Q.E.D.

**Lemma 4.4.5.**  $F^{l+1}$  is a surface generated by an abstract tree  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^{l+1}$ .

**Proof.** By Lemmas 4.4.1-4.4.4, we only need to prove that  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^{l+1}$  is an abstract tree.

By Lemma 4.3.10,  $\partial e_\gamma^l = \partial e_\gamma^{l+1}$  for  $\gamma \in m(l)$  and  $\partial e_{m^{l+1}}^{l+1} = \partial d_{j_0}^l$ . Let  $\{e_\gamma^* \mid \gamma \in m(l+1)\}$  be a set of pairwise disjoint arcs obtained by pushing  $inte_\gamma^{l+1}$  off  $\cup_f E_f^l$  in  $\partial \mathcal{V}_- \times I$ . By Proposition 4(1) for  $k = l$ , each component of  $\cup_{f=0}^l E_f^l \cup_{\gamma \in m(l)} e_\gamma^*$  is a tree. By Definition 4.3.3,  $E_f^l = E_f^{l+1}$  for  $1 \leq f \leq l$ , and  $E_0^{l+1}, E_l^{l+1} \subset E_0^l$ . Hence each component of  $\cup_{f=0}^{l+1} E_f^{l+1} \cup_{\gamma \in m(l)} e_\gamma^*$  is also a tree. Note that  $m^{l+1} = j_0$ .

By Lemma 3.3.1,  $d_{j_0}^l = \cup_{i=\delta(0)}^{\theta(0)} d_{i,0} \cup_{i=\delta(0)}^{\theta(0)} e_{i,0}$ . Since  $d_{j_0}^l$  intersects  $c_{l+1}^l$  in one point lying

in  $intd_{0,0} \subset E_0^l$ , and  $E_0^{l+1} \cup E_{l+1}^{l+1} = E_0^l - c_{l+1}^l \times (-1, 1)$ ,  $\cup_{f=0}^{l+1} E_f^{l+1} \cup_{\gamma \in m(l)} e_\gamma^*$  contains at least two components. Furthermore,  $E_0^{l+1}$  and  $E_{f_{\delta(0),0}}^{l+1}$  lies in one component of  $\cup_{f=0}^{l+1} E_f^{l+1} \cup_{\gamma \in m(l)} e_\gamma^*$ ,  $E_{l+1}^l$  and  $E_{f_{\theta(0),0}}^{l+1}$  lies in another component of  $\cup_{f=0}^{l+1} E_f^{l+1} \cup_{\gamma \in m(l)} e_\gamma^*$  as in Figure 18. Since  $\partial_1 e_{m^{l+1}}^{l+1} = \partial_1 d_{j_0}^l \subset E_{f_{\delta(0),0}}^{l+1}$  and  $\partial_2 e_{m^{l+1}}^{l+1} = \partial_2 d_{j_0}^l \subset E_{f_{\theta(0),0}}^{l+1}$ , so each component of  $\cup_{f=0}^{l+1} E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^*$  is also an tree. By Definition 2.1.2,  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^{l+1}$  is an abstract tree. Q.E.D.

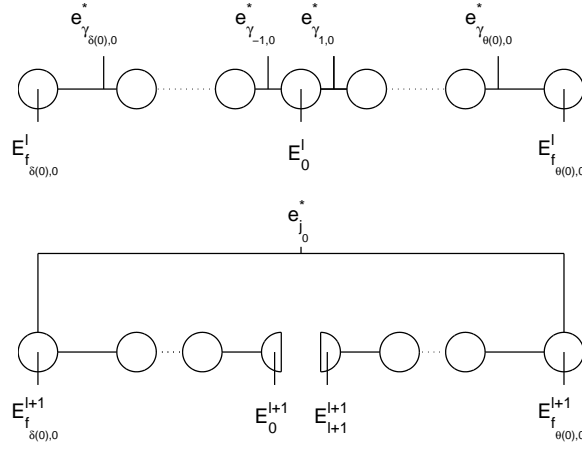


Figure 18

## 4.5 The properties of $d_j^{l+1}$

**Lemma 4.5.1.** Suppose that  $j \in \{1, \dots, n\} - m(l+1)$  and  $\lambda \in m(l+1)$ . If  $j < \lambda$ , then either  $d_j^{l+1}$  is disjoint from  $e_\lambda^{l+1} \times I$  or  $j \in I(w_\lambda, l+1)$  and  $s(w_\lambda) = -$ .

**Proof.** By assumption, Proposition 4 holds for  $k = l$ .

Suppose that  $j < \lambda$  and  $j \notin I(w_\lambda, l+1)$ . Then, by Proposition 4(3),  $d_j^l$  is disjoint from  $e_\lambda^l \times I$  for  $\lambda \in m(l)$ . Now there are two cases:

Case 1.  $j < m^{l+1} = j_0$ .

Suppose that  $\lambda = m^{l+1}$ . Since  $j \notin I(w_{m^{l+1}}, l+1)$ , by Lemma 4.1.5, Lemma 4.3.7,  $d_j^{l+1} = d_j^l$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I \subset \partial W_{j_0}^l \times I$ .

Suppose  $\lambda \in m(l)$ . Since  $j < j_0$ , by Lemma 4.3.5(3),  $d_j^{l+1} = d_j^l$  is disjoint from  $(a_{0,\alpha}^* \cup b_{0,\alpha}^*) \times I$ . Hence  $d_j^{l+1}$  is disjoint from  $b_\alpha \times I$ . By Definition 4.3.9,  $e_\lambda^{l+1} \times I \subset e_\lambda^l \times I \cup_\alpha b_\alpha \times I$ . Hence  $d_j^{l+1}$  is disjoint from  $e_\lambda^{l+1} \times I$ .



Case 2.  $j > j_0 = m^{l+1}$ .

Since  $\lambda > j$ ,  $\lambda > j_0$ . By Lemma 4.1.5,  $d_j^l, e_\lambda^l \times I$  are disjoint from  $\partial W_{j_0}^l \times I$ . If  $j < \gamma_\alpha$ , then, by Lemma 3.5.3,  $d_j^l$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$  except  $j = j_\alpha$ . Now there are two sub-cases:

(1)  $j \neq j_\alpha$  for each  $j_\alpha \in L(c_{l+1}^l)$ .

By Lemma 4.3.7(2),  $d_j^{l+1} = (d_j^l - \cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha < j} H_\alpha(d_j^l \cap (a_\alpha \times I)_{\gamma_\alpha})$ . By Construction(\*) (3), (4) and Lemma 4.3.5(3),  $d_j^l - \cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}$  is disjoint from  $(a_{0,\alpha}^* \cup b_{0,\alpha}^*) \times I$ . By Lemma 4.3.10 and Lemma 4.4.3,  $e_\lambda^{l+1} \times I = (e_\lambda^l \times I - \cup_{\gamma_\alpha \leq \lambda} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\gamma_\alpha \leq \lambda} H_\alpha(e_\lambda^l \times I \cap (a_\alpha \times I)_{\gamma_\alpha})$ . By Lemma 4.3.5(4),  $e_\lambda^l \times I - \cup_{\gamma_\alpha \leq \lambda} (a_\alpha \times I)_{\gamma_\alpha}$  is disjoint from  $(a_{0,\alpha}^* \cup b_{0,\alpha}^*) \times I$ . By Lemma 4.3.10(2),  $d_j^{l+1} \cap e_\lambda^{l+1} \times I = S_1 \cup S_2$  where  $S_1 = (d_j^l - \cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}) \cap (e_\lambda^l \times I - \cup_{\gamma_\alpha \leq \lambda} (a_\alpha \times I)_{\gamma_\alpha})$  and  $S_2 = H(e_\lambda^l \times I \cap (\cup_{\gamma_\alpha \leq \lambda} (a_\alpha \times I)_{\gamma_\alpha})) \cap H(d_j^l \cap (\cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}))$ . Since  $H$  is injective, by Proposition 4(3) for  $k = l$ , the lemma holds.

(2)  $j = j_\beta$  for some  $j_\beta \in L(c_{l+1}^l)$ .

Now  $(a_\beta \times \{0\})_{\gamma_\beta} = a_\beta \subset d_j^l$ . By Lemma 4.3.10 and Lemma 4.4.3,  $d_j^{l+1} = (d_j^l - a_\beta \cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha}) \cup H(d_j^l \cap (\cup_{\gamma_\alpha < j} (a_\alpha \times I)_{\gamma_\alpha})) \cup b_\beta$ . By assumption,  $\lambda \neq \gamma_\beta$ . Hence, by the same argument in (1), the lemma holds. Q.E.D.

**Lemma 4.5.2.** If  $j > \lambda$ , then each component of  $d_j^{l+1} \cap e_\lambda^{l+1} \times I$  is a core of  $e_\lambda^{l+1} \times (0, 1)$ .

**Proof.** By Proposition 4(2) for  $k = l$  and the same argument as that in the proof of Lemma 4.4.4, the lemma holds. Q.E.D.

**Lemma 4.5.3.** For each  $j \notin L(c_{l+1}^l)$ ,  $d_j^{l+1} = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}^*$  satisfying the following conditions:

- (1)  $\gamma_{i,j} \in m(l)$  is as in Lemma 3.1.2, and  $e_{\gamma_{i,j}}^*$  is a core of  $e_{\gamma_{i,j}}^{l+1} \times (0, 1)$  for  $1 \leq i \leq \theta(j) - 1$ .
- (2)  $f_{i,j}$  is as in Lemma 3.1.2. Furthermore, if  $f_{i,j} = 0$ , then  $d_{j,0}^l$  is either properly embedded in  $E_0^{l+1}$  or  $E_{l+1}^{l+1}$  which is disjoint from  $\cup_{\gamma < j} \text{inte}_\gamma^{l+1} \times I$ , if  $f_{i,j} \neq 0$ , then  $d_{j,f_{i,j}}^l$  is properly embedded in  $E_{f_{i,j}}^{l+1}$  which is disjoint from  $\cup_{\gamma < j} \text{inte}_\gamma^{l+1} \times I$  for  $1 \leq i \leq \theta(j)$ .
- (3)  $d_j^{l+1}$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e_\gamma^{l+1} \times I$ .

**Proof.** By Lemma 3.1.2, in  $F^l$ ,  $d_j^l = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}^l$  such that  $\gamma_{i,j} < j$  and  $d_{j,f_{i,j}}^l$  is disjoint from  $\cup_{\gamma < j} \text{inte}_\gamma^l \times I$ . Since  $j \notin L(c_{l+1}^l)$ ,  $d_{j,f}^l$  is disjoint from  $c_{l+1}^l \times I$  even if  $f = 0$ .

Hence  $d_{j,0}^l$  lies in one of  $E_0^{l+1}$  and  $E_{l+1}^{l+1}$ .

Suppose that  $\gamma < j$ . There are two cases:

(1)  $j < j_0$ .

Now  $\gamma_{i,j} < j < j_0$ . By Lemma 4.3.7 and Definition 4.3.9,  $e_{\gamma_{i,j}}^{l+1} \times I = e_{\gamma_{i,j}}^l \times I$  and  $d_j^{l+1} = d_j^l$ . Furthermore,  $e_{\gamma}^{l+1} \times I = e_{\gamma}^l \times I$  for  $\gamma < j$ . In this case,  $e_{\gamma_{i,j}}$  is also a core of  $e_{\gamma_{i,j}}^{l+1} \times (0, 1)$ . By Lemma 3.1.2, the lemma holds.

(2)  $j > j_0$ .

By Lemma 4.1.5,  $d_{j,f_{i,j}}^l \subset d_j^l$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I \subset \partial W_{j_0}^l \times I$ .

Now we claim that  $\text{int}d_{j,f_{i,j}}^l$  is disjoint from  $(a_{0,\alpha}^* \cup b_{0,\alpha}^*) \times I$ . By Construction(\*) (3) and (4), if  $\text{int}d_{j,f_{i,j}}^l \cap (a_{0,\alpha}^* \cup b_{0,\alpha}^*) \times I \neq \emptyset$ , then  $\text{int}d_{j,f_{i,j}}^l \cap (a_{0,\alpha}^* \cup b_{0,\alpha}^*) \times I \subset (a_{\lambda} \times I)_{\gamma_{\lambda}}$  for some  $j_{\lambda} \in L(c_{l+1}^l)$ . By assumption,  $j \neq j_{\lambda}$ . By Lemma 4.3.5,  $j > \gamma_{\lambda}$ , contradicting Lemma 3.1.2.

If  $\gamma < j$ , then, by Lemma 3.1.2,  $d_{j,f_{i,j}}^l$  is disjoint from  $\text{int}e_{\gamma}^l \times I$ . By Definition 4.3.9,  $\text{int}e_{\gamma}^{l+1} \times I \subset \cup_{\alpha} b_{\alpha} \times I \cup \text{int}e_{\gamma}^l \times I$ . Hence  $d_{j,f_{i,j}}^l$  is disjoint from  $\text{int}e_{\gamma}^{l+1} \times I$ . Now  $d_j^{l+1} = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}^*$  where  $e_{\gamma_{i,j}}^* = (e_{\gamma_{i,j}} - e_{\gamma_{i,j}} \cap (\cup_{\alpha} (a_{\alpha} \times I)_{\gamma_{\alpha}})) \cup \cup_{\alpha} H_{\alpha}(e_{\gamma_{i,j}} \cap (a_{\alpha} \times I)_{\gamma_{\alpha}})$  is a core of  $e_{\gamma_{i,j}}^{l+1} \times (0, 1)$ . Now each component of  $d_j^{l+1} \cap (\cup_{\gamma < j} e_{\gamma}^{l+1} \times I)$  is  $e_{\gamma_{i,j}}^*$  for  $1 \leq i \leq \theta(j) - 1$ . By Lemma 4.5.1 and 4.5.2, (1) and (2) holds.

By Lemma 4.3.7,  $\partial d_j^{l+1} = \partial d_j^l$ . By Lemma 4.3.10,  $(\partial e_{\gamma}^{l+1}) \times I = (\partial e_{\gamma}^l) \times I$  for  $\gamma \in m(l)$ . Hence  $\partial d_j^{l+1}$  is disjoint from  $e_{\gamma}^{l+1} \times I$ . Since  $j > j_0$ ,  $j \notin I(w_{j_0}, l)$ , by Lemma 4.1.5(4),  $\partial d_j^{l+1} = \partial d_j^l$  is disjoint from  $e_{m^{l+1}}^{l+1} \times I \subset \partial W_{j_0}^l \times I$ . By Lemma 4.3.7(2) and Definition 4.3.9,  $d_j^l$  is properly embedded in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e_{\gamma}^{l+1} \times I$ . By Lemma 3.1.2,  $f_{i,j} \neq f_{i,j}$  and  $\gamma_{i,j} = \gamma_{r,j}$  for  $i \neq r$ . By Lemma 4.5.2,  $d_j^{l+1}$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e_{\gamma}^{l+1} \times I$ . Q.E.D.

**Lemma 4.5.4.** If  $\gamma < j_{\alpha}$ , then  $\text{inta}_{0,\alpha}^* \cup \text{int}b_{0,\alpha}^*$  is disjoint from  $e_{\gamma}^{l+1} \times I$ .

**Proof.** Suppose that  $\gamma < j_{\alpha}$ . There are three cases:

Case 1.  $\gamma < j_0 = m^{l+1}$ .

By Definition 4.3.9,  $e_{\gamma}^{l+1} \times I = e_{\gamma}^l \times I$ . By Lemma 4.3.5(4), the lemma holds.

Case 2.  $\gamma = j_0 = m^{l+1}$ .

By Lemma 3.5.2 and Definition 4.3.9,  $a_{\alpha}^0 \subset d_{j_0}^l$ . Hence  $e_{m^{l+1}}^{l+1} \times I \subset (\partial W_{j_0}^l - \text{inta}_{\alpha}^0) \times I$ .

By Lemma 4.1.5(8) and Construction(\*) (3) and (4), the lemma holds.

Case 3.  $\gamma > j_0 = m^{l+1}$ . By Lemma 4.3.5(4),  $e_\gamma^l \times I \cap (inta_{0,\alpha}^* \cup intb_{0,\alpha}^*) \subset \cup_{\beta \leq \gamma} (a_\beta \times I)_{\gamma_\beta}$ . Note that  $\gamma_\alpha > j_\alpha$  if  $\gamma_\alpha \neq \emptyset$ . Since  $\gamma < j_\alpha$ ,  $\gamma < \gamma_\alpha$  if  $\gamma_\alpha \neq \emptyset$ .

By Lemma 4.3.10(2) and (3) and Lemma 4.4.3,  $e_\gamma^{l+1} \times I = (e_\gamma^l \times I - (e_\gamma^l \times I \cap (\cup_{\gamma_\beta \leq \gamma} (a_\beta \times I)_{\gamma_\beta}))) \cup_{\gamma_\beta \leq \gamma} H_\beta(e_\gamma^l \times I \cap (a_\beta \times I)_{\gamma_\beta})$ . Note that  $H_\beta(e_\gamma^l \times I \cap (a_\beta \times I)_{\gamma_\beta}) \subset b_\beta \times I$ . By Lemma 4.3.5,  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ . Hence the lemma holds. Q.E.D.

**Lemma 4.5.5.** Suppose that  $j_\alpha \in L(c_{l+1}^l)$  and  $\alpha \neq 0$ . Then

- (1) Each component of  $d_{j_\alpha}^{l+1} \cap (\cup_{\gamma < j_\alpha} e_\gamma^{l+1} \times I)$  is a core of  $e_\gamma^{l+1} \times I$  for  $\gamma \in \{m^{l+1}\} \cup \{\gamma_{i,\alpha} \mid i < \delta(0, \alpha) \text{ or } i > \theta(0, \alpha)\} \cup \{\gamma_{i,0} \mid i < \delta(0, \alpha) \text{ or } i > \theta(0, \alpha)\}$ .
- (2) Each component of  $d_{j_\alpha}^{l+1} - \cup_{\gamma < j_\alpha} int e_\gamma^{l+1} \times I$  lies in  $E_f^{l+1}$  for  $f \in \{0, l+1\} \cup \{f_{i,\alpha} \mid i \leq \delta(0, \alpha) \text{ or } i \geq \theta(0, \alpha)\} \cup \{f_{i,0} \mid i < \delta(0, \alpha) \text{ or } i > \theta(0, \alpha)\}$ .
- (3)  $d_{j_\alpha}^{l+1}$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j_\alpha} e_\gamma^{l+1} \times I$ .

**Proof.** Since  $a_\alpha = a_\alpha \times \{0\} \subset (a_\alpha \times I)_{\gamma_\alpha}$ . By Lemma 4.3.7(3),  $d_{j_\alpha}^{l+1} = S \cup b_\alpha$  where  $S = (d_{j_\alpha}^l - a_\alpha \cup_{\gamma_\beta < j_\alpha} (a_\beta \times I)_{\gamma_\beta}) \cup_{\gamma_\beta < j_\alpha} H_\beta(d_{j_\alpha}^l \cap (a_\beta \times I)_{\gamma_\beta})$ .

By Definition 3.5.1,  $a_\alpha = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$ . By Lemma 3.3.1,  $d_{j_\alpha}^l - int a_\alpha = \cup_{i=\delta(0,\alpha)}^{\delta(0,\alpha)-1} (d_{i,\alpha} \cup e_{i,\alpha}) \cup_{i=\theta(0,\alpha)+1}^{\theta(\alpha)} (d_{i,\alpha} \cup e_{i,\alpha})$ . By the argument in Lemma 4.5.3,  $S_1 = \cup_{i=\delta(0,\alpha)}^{\delta(0,\alpha)-1} (d_{i,\alpha} \cup e_{i,\alpha}^*) \cup_{i=\theta(0,\alpha)+1}^{\theta(\alpha)} (d_{i,\alpha} \cup e_{i,\alpha}^*)$  satisfying the following conditions:

- (1)  $d_{i,\alpha}$  is an arc in  $E_{f_{i,\alpha}}^{l+1}$  such that  $int d_{i,\alpha}$  is disjoint from  $e_\gamma^{l+1} \times I$  for  $\gamma < j_\alpha$ .
- (2)  $e_{i,\alpha}^* = (e_{i,\alpha} - e_{i,\alpha} \cap (\cup_\alpha (a_\alpha \times I)_{\gamma_\alpha})) \cup \cup_\alpha H_\alpha(e_{i,\alpha} \cap (a_\alpha \times I)_{\gamma_\alpha})$  is a core of  $e_{\gamma_{i,\alpha}}^{l+1} \times I$ .

By the construction,  $b_\alpha = b_\alpha \times \{0\} = a_{0,\alpha}^* \cup (\partial W_{j_0}^l - a_\alpha^0) \times \{t_\alpha\} \cup b_{0,\alpha}^*$ . Note that  $a_{0,\alpha}^* \subset E_{f_{\delta(0,\alpha),\alpha}}^{l+1}$  and  $b_{0,\alpha}^* \subset E_{f_{\theta(0,\alpha),\alpha}}^{l+1}$ . Let  $S_2 = (\partial W_{j_0}^l - a_\alpha^0) \times \{t_\alpha\}$ . Note that  $\gamma_{i,0} < j_0$  for  $\delta(0) \leq i \leq \theta(0)$ . By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_{\gamma_{i,0}}^l \times I$ . By Lemma 3.3.1,  $d_{j_0}^l = \cup_{\delta(0)}^{\theta(0)} d_{i,0} \cup_{\delta(0)}^{\theta(0)} e_{i,0}$ . By Lemma 3.5.1,  $a_\alpha^0 = \cup_{\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,0} \cup_{\delta(0)}^{\theta(0)} e_{i,0}$ . Hence  $S_2$  intersects  $e_\gamma^{l+1} \times I$  in a core of  $e_\gamma^{l+1} \times I$  for  $\gamma \in \{\gamma_{i,0} \mid i < \delta(0, \alpha) \text{ or } i > \theta(0, \alpha)\}$ , and  $E_{f_{i,\alpha}}^{l+1}$  in an arc  $d_{i,0} \times \{t_\alpha\}$  for  $f \in \{f_{i,0} \mid i < \delta(0, \alpha) \text{ or } i > \theta(0, \alpha)\}$ . In particular, by Definition 4.3.9,  $S_2$  intersects  $e_{m^{l+1}}^{l+1} \times I$  in a core of  $e_{m^{l+1}}^{l+1} \times (0, 1)$ . By Lemma 4.5.4,  $a_{0,\alpha}^* \cup b_{0,\alpha}^*$  is disjoint from  $int e_\gamma^{l+1} \times I$  for  $\gamma < j_\alpha$ .

Now we claim that  $d_{i,0} \times \{t_\alpha\}$  is disjoint from  $int e_\gamma^{l+1} \times I$  for  $\gamma < j_\alpha$ . There are two cases:

Case 1.  $\gamma \leq j_0 = m^{l+1}$ .

Suppose that  $\gamma < j_0$ . By Definition 4.3.9,  $e_\gamma^{l+1} \times I = e_\gamma^l \times I$ . By Lemma 4.1.5(6),  $d_{i,0} \times \{t_\alpha\} \subset d_{i,0} \times I$  is disjoint from  $e_\gamma^{l+1} \times I$ . Since  $d_{i,0} \subset d_{j_0}^l$ ,  $inte_{m^{l+1}}^{l+1} \times I = (\partial W_{j_0}^l - d_{j_0}^l) \times I$  is disjoint from  $d_{i,0} \times \{t_\alpha\}$ .

Case 2.  $\gamma > j_0$ .

Since  $\gamma < j_\alpha$ ,  $\gamma < \gamma_\alpha$ . Since  $\gamma > j_0 > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , by Lemma 3.5.3,  $e_\gamma^l \times I$  is disjoint from  $(inta_\alpha \times I)_{\gamma_\alpha}$ .

By Lemma 4.3.10,  $e_\gamma^{l+1} \times I = (e_\gamma^l \times I - \cup_{\gamma_\beta \leq \gamma} (a_\beta \times I)_{\gamma_\beta}) \cup_{\gamma_\beta \leq \gamma} H_\beta(e_\gamma^l \times I \cap (a_\beta \times I)_{\gamma_\beta})$ . Note that  $H_\beta(e_\gamma^l \times I \cap (a_\beta \times I)_{\gamma_\beta}) \subset b_\beta \times I$ . By Lemma 4.3.5,  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ . Hence  $e_\gamma^{l+1} \times I$  is disjoint from  $H_\alpha((inta_\alpha \times I)_{\gamma_\alpha}) = intb_\alpha \times I$ . Thus  $d_{i,0} \times \{t_\alpha\}$  is disjoint from  $inte_\gamma^{l+1} \times I$ .

By Lemma 3.3.4(3),  $\{\gamma_{i,\alpha} \mid i > \theta(0, \alpha) \text{ or } i < \delta(0, \alpha)\} \cap \{\gamma_{i,0} \mid i > \theta(0, \alpha) \text{ or } i < \delta(0, \alpha)\} = \emptyset$ . By Lemma s 4.5.1 and 4.5.2,  $d_j^{l+1}$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e_\gamma^{l+1} \times I$ . Q.E.D.

## 4.6 The proofs of Propositions 4-6 for the case: $k = l + 1$ and

$$s(v_{l+1}) = +$$

In this section, we shall finish the proofs of Propositions 4-6 for the case:  $k = l + 1$  and  $s(v_{l+1}) = +$ .

**Lemma 4.6.1.**  $\partial c_i^{l+1}$  is disjoint from  $d_j^{l+1}$  and  $e_\gamma^{l+1}$ .

**Proof.** By Definition 4.3.3,  $c_i^{l+1} = c_i^l$ . By Lemma 4.1.5(5),  $\partial c_i^{l+1}$  is disjoint from  $\partial W_{j_0}^l \times I$ . By Lemma 3.5.8,  $\partial c_i^{l+1}$  is disjoint from  $b_\alpha \times I$ . By Proposition 4(5),  $\partial c_i^{l+1}$  is disjoint from  $d_j^l$  and  $e_\gamma^l \times I$ . By Definition 4.3.9,  $d_j^{l+1} \subset d_j^l \cup \partial W_{j_0}^l \times I \cup_\alpha b_\alpha \times I$  and  $e_\gamma^{l+1} \times I \subset e_\gamma^l \times I \cup \partial W_{j_0}^l \times I \cup_\alpha b_\alpha \times I$ . Hence the lemma holds. Q.E.D.

**The proof of Proposition 4.** By Lemmas 4.4.1-4.4.5, Proposition 4(1) holds. By Lemma 4.5.3(3) and Lemma 4.5.5(3),  $d_j^{l+1}$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e_\gamma^{l+1} \times I$ . Hence Proposition 4(2) holds. Proposition 4(3) follows from Lemma 4.5.1.

If  $j \notin L(c_{i+1}^l)$ , then, by Proposition 4(4),  $c_i^{l+1} = c_i^l$  intersects  $d_{j,f_{i,j}}^l$  in at most one point. By Lemma 4.5.3 and Lemma 4.3.7,  $d_j^{l+1} - \cup_{\gamma < j} inte_\gamma^{l+1} \times I$  intersects  $c_i^{l+1}$  in at most one

point.

Suppose that  $j = j_\alpha$ . By Lemma 3.3.4(4),  $\{f_{i,\alpha} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} \cap \{f_{i,\beta} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} = \emptyset$ .

Suppose first that  $\delta(0, \alpha) = \theta(0, \alpha)$ , then, by Lemma 3.5.8,  $a_{0,\alpha}^* \subset E_0^{l+1} \subset b_{0,\alpha}^* \subset E_{l+1}^{l+1}$ . Note that  $E_0^{l+1}, E_{l+1}^{l+1} \subset E_0^l$ . By Lemma 3.3.1,  $\{0, l+1\} \cap (\{f_{i,\alpha} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} \cup \{f_{i,\beta} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\}) = \emptyset$ .

Suppose now that  $\delta(0, \alpha) \neq \theta(0, \alpha)$ . Then, by Lemma 3.3.1,  $\{f_{\delta(0,\alpha),\alpha}, f_{\theta(0,\alpha),\alpha}\} \cap (\{f_{i,\alpha} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\} \cup \{f_{i,\beta} \mid i > \theta(\alpha, \beta) \text{ or } i < \delta(\alpha, \beta)\}) = \emptyset$ .

By Lemma 4.5.5,  $d_j^{l+1} - \cup_{\gamma < j} \text{inte}_\gamma^{l+1} \times I = \cup_{i=\delta(\alpha)}^{\delta(0,\alpha)-1} d_{i,\alpha} \cup_{i=\theta(0,\alpha)+1}^{\theta(\alpha)} d_{i,\alpha} \cup_{i=\delta(0)}^{\delta(0,\alpha)-1} d_{i,0} \times \{t_\alpha\} \cup_{i=\theta(0,\alpha)+1}^{\theta(0)} d_{i,0} \times \{t_\alpha\} \cup a_{0,\alpha}^* \cup b_{0,\alpha}^*$ . By Lemma 4.3.7, there is at most one component  $b^i$  of  $d_j^{l+1} - \cup_{\gamma < j} \text{inte}_\gamma^{l+1} \times I$  such that  $b^i \cap c^{l+1} \neq \emptyset$  for each  $i \geq l+2$ .

By Lemma 4.1.5(6), Construction(\*) (3), (4) and Lemma 3.5.8,  $c_i^{l+1} = c_i^l$  intersects  $a_{0,\alpha}^*$  in at most one point. Furthermore,  $c_i^{l+1} = c_i^l$  intersects  $a_{0,\alpha}^*$  in one point if and only if  $c_i^{l+1} \cap a_{0,\alpha}$  in one point, where  $a_{0,\alpha}$  is as in Lemma 3.5.5. Similarly,  $c_i^{l+1} = c_i^l$  intersects  $b_{0,\alpha}^*$  in one point if and only if  $c_i^{l+1} \cap b_{0,\alpha}$  in one point. By Lemma 4.1.5(6),  $c_i^{l+1}$  intersects  $d_{i,0} \times \{t_\alpha\}$  in one point if and only if  $c_i^l$  intersects  $d_{i,0}$  in one point. By Proposition 4(4) and Lemma 3.3.1,  $c_i^{l+1} = c_i^l$  intersects  $d_{i,\alpha}$  in at most one point. Hence Proposition 4(4) holds.

Proposition 4(5) follows from Lemma 4.6.1. Q.E.D.

**The proof of Proposition 5.** Suppose that  $j \notin L(c_{l+1}^l)$ . By the argument in the proof of Lemma 4.5.3 and Lemma 3.1.4,  $L(d_j^{l+1}) = L(d_j^l)$ .

Suppose that  $j = j_\alpha \in L(c_{l+1}^l)$ . By the argument in the proofs of Lemma 4.5.5 and Proposition 4 for  $k = l+1$ ,  $L(d_j^{l+1}) = \cup_{i=\delta(\alpha)}^{\delta(0,\alpha)-1} L(d_{i,\alpha}) \cup_{i=\theta(0,\alpha)+1}^{\theta(\alpha)} L(d_{i,\alpha}) \cup_{i=\delta(0)}^{\delta(0,\alpha)-1} L(d_{i,0}) \cup_{i=\theta(0,\alpha)+1}^{\theta(0)} L(d_{i,0}) \cup L(a_{0,\alpha}^*) \cup L(b_{0,\alpha}^*)$ .

Now there are two cases:

Case 1.  $\delta(0, \alpha) = \theta(0, \alpha) = 0$ .

By Lemma 3.5.8(1),  $L(a_{0,\alpha}^*) \cup L(b_{0,\alpha}^*) = L(d_{0,0}) \cup L(d_{0,\alpha}) - L(d_{0,0}) \cap L(d_{0,\alpha})$ . By Lemma 3.3.1,  $f_{i,0} \neq f_{r,\alpha}$ . By Proposition 4(4),  $L(d_{i,0}) \cap L(d_{i,\alpha}) = \emptyset$  for  $i \neq 0$ . Hence,  $L(d_j^{l+1}) = L(d_j^l) \cup L(d_{m^{l+1}}^l) - L(d_j^l) \cap L(d_{m^{l+1}}^l)$ .

Case 2.  $\delta(0, \alpha) \neq \theta(0, \alpha)$ .

By Lemma 3.3.4,  $L(d_{i,0}) = L(d_{i,\alpha})$  for  $\delta(0, \alpha) < i < \theta(0, \alpha)$ . By Lemma 3.5.8(2),  $L(a_{0,\alpha}^*) = L(d_{\delta(0,\alpha),0}) \cup L(d_{\delta(0,\alpha),\alpha}) - L(d_{\delta(0,\alpha),0}) \cap L(d_{\delta(0,\alpha),\alpha})$ . Similarly,  $L(b_{0,\alpha}^*) = L(d_{\theta(0,\alpha),0}) \cup L(d_{\theta(0,\alpha),\alpha}) - L(d_{\theta(0,\alpha),0}) \cap L(d_{\theta(0,\alpha),\alpha})$ . Hence  $L(d_j^{l+1}) = L(d_j^l) \cup L(d_{m^{l+1}}^l) - L(d_j^l) \cap L(d_{m^{l+1}}^l)$ . Q.E.D.

**Lemma 4.6.3.**  $b_j^{l+1}$  is obtained by doing band sums with some copies of  $\partial W_{j_0}^l$  to  $d_j^l$ .

**Proof.** Since each component of  $d_j^l \cap (a_\alpha \times I)_{\gamma_\alpha}$  is a core of  $(a_\alpha \times I)_{\gamma_\alpha}$ , say  $a_\alpha \times \{t_j\}$ . By Definition 4.3.6,  $H_\alpha(a_\alpha \times \{t_j\})$  intersects  $a_{0,\alpha}^* \times I$  in  $a_{0,\alpha}^* \times \{t_j\}$ ,  $b_{0,\alpha}^* \times I$  in  $b_{0,\alpha}^* \times \{t_j\}$ ,  $(\partial W_{j_0}^l - \text{inta}_\alpha^0) \times I$  in  $(\partial W_{j_0}^l - \text{inta}_\alpha^0) \times \{t_j\}$ . Now by Lemma 3.5.5,  $a_{0,\alpha}^* \times \{t_j\} \cup b_{0,\alpha}^* \times \{t_j\} \cup a_\alpha \times \{t_j\} \cup a_\alpha^0 \times \{t_j\}$  bounds a disk in  $F^{l+1}$ . Hence, the lemma holds. Q.E.D.

**The Proof of Proposition 6.** Suppose that  $i \geq l+2$  and  $s(v_i) = -$ . Then, by Proposition 6, there is a properly embedded disk  $V_i^l$  in  $\mathcal{V}_-$  such that  $\partial V_i^l \cap F^l = c_i^l \cup_{r \in I(v_i, l)} c_r^l$ . Now we denote by  $V_i^{l+1}$  the disk  $V_i^l$ . By Lemma 4.1.5(7),  $\partial W_{j_0}^l \times I$  is disjoint from  $\partial V_i^l - c_i^l \cup_{r \in I(v_i, l)} v_r^l$  for each  $i \geq l+2$  with  $s(v_i) = -$ . Since  $F^{l+1} = (F^l - c_{l+1}^l \times (-1, 1)) \cup e_{m^{l+1}}^{l+1} \times I$ ,  $\partial V_i^{l+1} \cap F^{l+1} = c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_r^{l+1}$ . In this case, if  $l+1 \in I(v_i, l)$ , then  $I(v_i, l+1) = I(v_i, l) - \{l+1\}$ , if not, then  $I(v_i, l+1) = I(v_i, l)$ .

Now suppose that  $j \notin m(l+1)$  and  $s(w_j) = -$ . By Lemma 4.1.5(7),  $\partial W_{j_0}^l \times I$  is disjoint from  $\partial W_j^l$ . By Proposition 6,  $\partial W_j^l \cap F^l = d_j^l \cup_{r \in I(w_j, l)} d_r^l$ . Now let  $C_j = (\partial W_j^l - d_j^l \cup_{r \in I(w_j, l)} d_r^l) \cup d_j^{l+1} \cup_{i \in I(w_j, l)} d_i^{l+1}$ . Then  $C_j \cap F^{l+1} = d_j^{l+1} \cup_{r \in I(w_j, l)} d_r^{l+1}$ . Since  $s(w_{j_0}) = -$ , by Lemma 2.2.4 and Lemma 4.1.3,  $j_0 = m^{l+1} \notin I(w_j, l)$ . Hence  $I(w_j, l+1) = I(w_j, l)$ . By Lemma 4.6.3,  $C_j$  is obtained by doing band sums with copies of  $\partial W_{j_0}^l$  to  $\partial W_j^l$ . Hence  $C_j$  bounds a disk in  $\mathcal{W}_-$ , denoted by  $W_j^{l+1}$ .

Now by Proposition 6,  $(\partial V_i - c_i^l \cup_{r \in I(v_i, l)} c_r^l) \cap (\partial W_j^l - d_j^l \cup_{r \in I(w_j, l)} d_r^l) = \emptyset$  for  $i \geq l+2$  with  $s(v_i) = -$  and  $j \notin m(l)$  with  $s(w_j) = -$ . Now there are two case:

(1)  $l+1 \notin I(v_i, l)$ .

Now  $I(v_i, l+1) = I(v_i, l)$ . Hence  $V_i^{l+1} \cap W_j^{l+1} = (c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_r^{l+1}) \cap (d_j^{l+1} \cup_{r \in I(w_j, l+1)} d_r^{l+1})$ .

(2)  $l+1 \in I(v_i, l)$ .

Now  $I(v_i, l+1) = I(v_i, l) - \{l+1\}$ . By Proposition 6(2),  $c_{l+1}^l$  is disjoint from  $W_j^l - d_j^l \cup_{r \in I(w_j, l)} d_r^l$ . Obviously,  $c_{l+1}^l$  is disjoint from  $d_j^{l+1}$  and  $d_r^{l+1}$ . Note that  $(\partial V_i^l - c_i^l \cup_{r \in I(v_i, l)} c_r^l) \cup c_{l+1}^l = \partial V_i^{l+1} - c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_r^{l+1}$ . Hence  $V_i^{l+1} \cap W_j^{l+1} = (c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_r^{l+1}) \cap d_j^{l+1} \cup_{r \in I(w_j, l+1)} d_r^{l+1}$ . Q.E.D

## 5 The Proofs of Propositions 4-6 for the case: $k = l+1$ and $s(v_{l+1}) = -$

In this section, we shall Propositions 4-6 for the case:  $k = l+1$  and  $s(v_{l+1}) = -$ . In the following argument, we assume  $s(v_{l+1}) = -$ .

### 5.1 The element of constructions

Recalling  $m^{l+1} = \text{Min}L(c_{l+1}^l)$ , and  $L(c_{l+1}^l) = \{\dots, j_{-1}, j_0 = m^{l+1}, j_1, \dots\}$ .

**Lemma 5.1.1.** Suppose that  $m^{l+1} \neq \emptyset$ . Then

- (1)  $V_{l+1}^l \subset \mathcal{V}_-$  such that  $\partial V_{l+1}^l - c_{l+1}^l$  is disjoint from  $F^l$ .
- (2)  $\partial V_{l+1}^l$  intersects  $d_{j_0}^l$  in one point  $p = \text{int}c_{l+1}^l \cap \text{int}d_{0,0}^l$ .
- (3) If  $j < j_0 = m^{l+1}$ , then  $d_j^l$  is disjoint from  $\partial V_{l+1}^l$ ; if  $\gamma < j_0 = m^{l+1}$ , then  $e_\gamma^l \times I$  is disjoint from  $\partial V_{l+1}^l$ .
- (4)  $s(w_{m^{l+1}}) = +$ .

**Proof.** (1) Since  $s(v_{l+1}) = -$ , by Proposition 6,  $V_{l+1}^l \subset \mathcal{V}_-$  and  $\partial V_{l+1}^l \cap F^l = c_{l+1}^l \cup_{r \in I(v_{l+1}, l)} c_r^l$ . If  $r \in I(v_{l+1}, l)$ , then, by Lemma 2.2.4,  $r < l+1$ . By Definition 2.3.1,  $I(v_{l+1}, l) = \emptyset$ . Thus (1) holds.

(2) By (1),  $\partial V_{l+1}^l \cap d_{j_0}^l = c_{l+1}^l \cap d_{j_0}^l$ . By the proof of Lemma 4.1.2,  $c_{l+1}^l$  intersects  $d_{j_0}^l$  in one point lying in  $\text{int}d_{0,0}^l \cap \text{int}c_{l+1}^l$ .

(3) Since  $d_j^l, e_\gamma^l \times I \subset F^l$ . Hence  $d_j^l \cap \partial V_{l+1}^l = d_j^l \cap c_{l+1}^l$  and  $e_\gamma^l \times I \cap \partial V_{l+1}^l = e_\gamma^l \times I \cap c_{l+1}^l$ . By the minimality of  $m^{l+1}$  in  $L(c_{l+1}^l)$ , (3) follows from the proof of Lemma 4.1.2.

(4) Suppose that  $s(w_{m^{l+1}}) = -$ . Then, by Proposition 6,  $W_{m^{l+1}}^l \subset \mathcal{W}_-$ . By (1),  $W_{m^{l+1}}^l \cap V_{l+1}^l = c_{l+1}^l \cap (d_{m^{l+1}}^l \cup_{r \in I(w_{m^{l+1}}, l)} d_r^l)$ . Since  $r \in I(w_{m^{l+1}}, l)$ , by Lemma 2.2.4,

$r < j_0 = m^{l+1}$ . By (2) and (3),  $V_{l+1}^l$  intersects  $W_{m^{l+1}}^l$  in one point. Hence  $\mathcal{V}_- \cup \mathcal{W}_-$  is stabilized, a contradiction. Q.E.D.

**Lemma 5.1.2.** Suppose that  $j_0 = m^{l+1} \neq \emptyset$ . If  $j_0 \in I(w_{\gamma_0}, l)$  for some  $\gamma_0 \in m(l)$  with  $s(w_{\gamma_0}) = -$ , then there is a neighborhood of  $d_{j_0}^l$  in  $F^l$ , say  $d_{j_0}^l \times [-2.5, 2.5]$ , which satisfies the following conditions:

- (1)  $(d_{j_0}^l \times I)_{\gamma_0} = d_{j_0}^l \times [-1, 1]$ .
- (2) If  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , then each component of  $d_{j_0}^l \times [-2.5, 2.5] \cap e_\gamma^l \times I$  is  $c \times [-2.5, 2.5] \subset e_\gamma^l \times (0, 1)$  where  $c \subset \text{int}d_{j_0}^l$  is a core of  $e_\gamma^l \times (0, 1)$ .
- (3) If  $\gamma_0 > \gamma > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , then  $d_{j_0}^l \times [-2.5, 2.5]$  is disjoint from  $e_\gamma^l \times I$ .
- (4) If  $\gamma > \gamma_0$ , then  $e_\gamma^l \times I$  is disjoint from  $d_{j_0}^l \times ([1, 2.5] \cup [-2.5, -1])$ .
- (5) If  $j \neq j_0 < \gamma_0$ , then  $d_j^l$  is disjoint from  $d_{j_0}^l \times [-2.5, 2.5]$ , if  $j > \gamma_0$ , then  $d_j^l$  is disjoint from  $d_{j_0}^l \times ([1, 2.5] \cup [-2.5, -1])$ .
- (6)  $\text{int}d_{i,0} \times [-2.5, -1.5] \cup [1.5, 2.5]$  is disjoint from  $e_\gamma^l \times I$  for each  $\gamma \in m(l)$  and  $\delta(0) \leq i \leq \theta(0)$ ,  $\text{int}d_{i,0} \times [-2.5, 2.5]$  is disjoint from  $e_\gamma^l \times I$  for  $\gamma < j_0$ .
- (7)  $d_{j_0}^l \times [-2.5, 2.5]$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$  for  $\alpha \neq 0$ , and  $d_j^l \times [-2.5, 2.5] \cap D_{0,\alpha}^* = a_\alpha^0 \times [-2.5, 2.5] \cap D_{0,\alpha}^*$ .
- (8) For each  $i \geq l+1$ ,  $\partial c_i^l$  is disjoint from  $d_{j_0}^l \times [-2.5, 2.5]$ .

**Proof.** (1) is trivial.

(2) Since  $\gamma < j_0$ , by Proposition 4(2) and Definition 2.1.5, each component of  $d_{j_0}^l \cap e_\gamma^l \times I$  is a core of  $e_\gamma^l \times I$ , say  $c$ , which is a core of  $e_\gamma^l \times (0, 1)$ . Note that  $d_{j_0}^l \subset e_\gamma^l$ . By Proposition 4(1) and Definition 2.1.4, each component of  $(d_{j_0}^l \times I)_{\gamma_0} \cap e_\gamma^l \times I$  is  $(c \times I)_{\gamma_0} \subset e_\gamma^l \times (0, 1)$ .

(3) By Lemma 3.2.3 and Lemma 3.2.6,  $(d_{j_0}^l \times I)_{\gamma_0}$  is disjoint from  $e_\gamma^l \times I$  for  $\gamma_0 > \gamma > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

(4) If  $\gamma > \gamma_0 \in m(l)$ , then, by Proposition 4(1), each component of  $e_\gamma^l \times I \cap (d_{j_0}^l \times I)_{\gamma_0} = (c \times I)_\gamma \subset (d_{j_0}^l \times (0, 1))_{\gamma_0}$ .

(5) If  $j \neq j_0 < \gamma_0$ , then, by Proposition 4(3),  $d_j^l$  is disjoint from  $(d_{j_0}^l \times I)_{\gamma_0}$ . By Proposition 4(2) and Definition 2.1.5, if  $j > \gamma_0$ , then each component of  $d_j^l \cap (d_{j_0}^l \times I)_{\gamma_0}$  is  $c \subset (d_{j_0}^l \times (0, 1))_{\gamma_0}$ .



(6) If  $j_0 < \gamma$ , then either  $\gamma = \gamma_0$  or  $(d_{j_0}^l \times I)_{\gamma_0}$  is disjoint from  $e_\gamma^l \times I$  by Lemma 3.2.6. If  $j_0 > \gamma$ , then, by Lemma 3.3.1 and Definition 2.1.4,  $(intd_{i,0} \times I)_{\gamma_0}$  is disjoint from  $e_\gamma^l \times I$ . If  $\gamma > \gamma_0$ , then each component of  $e_\gamma^l \times I \cap (d_{j_0}^l \times I)_{\gamma_0} = (c \times I)_\gamma \subset (d_{j_0}^l \times (0, 1))_{\gamma_0}$ .

(7) By Lemma 3.2.6 and Lemma 3.5.7,  $(d_{j_0}^l \times I)_{\gamma_0}$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$  for  $\alpha \neq 0$ .

(8) By Proposition 4(5),  $\partial c_i^l$  is disjoint from  $(d_{j_0}^l \times I)_{\gamma_0} \subset e_{\gamma_0}^l \times I$ .

Now  $d_{j_0}^l \times [-2, 2]$  can be obtained by making  $(d_{j_0}^l \times I)_{\gamma_0}$  wide slightly. Hence Lemma 5.1.2 holds. Q.E.D.

**Lemma 5.1.3.** Suppose  $j_0 \neq \emptyset$ . If  $j_0 \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$  with  $s(w_\gamma) = -$ , there is a neighborhood of  $d_{j_0}^l$  in  $F^l$ , say  $d_{j_0}^l \times [-2.5, 2.5]$ , which satisfies the following conditions:

- 1) If  $j \neq j_0$ , then  $d_{j_0}^l \times [-2.5, 2.5]$  is disjoint from  $d_j^l$ .
- 2) If  $\gamma > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , then  $d_{j_0}^l \times [-2.5, 2.5]$  is disjoint from  $e_\gamma^l \times I$ .
- 3) If  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , then each component of  $d_{j_0}^l \times [-2.5, 2.5] \cap e_\gamma^l \times I$  is  $c \times [-2.5, 2.5] \subset e_\gamma^l \times (0, 1)$  where  $c \subset intd_{j_0}^l$  is a core of  $e_\gamma^l \times (0, 1)$ .
- 4)  $d_{i,0} \times [-2.5, 2.5]$  is disjoint from  $int e_\gamma^l \times I$  for each  $\gamma \in m(l)$ .
- (5)  $d_{i,0} \times [-2.5, 2.5]$  is disjoint from  $\partial c_i^l$  for each  $i \geq l + 1$ .
- (6)  $d_{j_0}^l \times [-2.5, 2.5]$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$  for  $\alpha$ , and  $d_j^l \times [-2.5, 2.5] \cap D_{0,\alpha}^* = a_\alpha^0 \times [-2.5, 2.5] \cap D_{0,\alpha}^*$ .

**Proof.** By Lemma 5.1.1(4),  $s(w_{m^{l+1}}) = +$ . Hence if  $j_0 = m^{l+1} \in I(w_\gamma, l)$  then  $s(w_\gamma) = -$ . By assumption,  $j_0 \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$ . Now (1) follows from Proposition 4. (2) follows from Lemma 3.2.3. (3) follows Proposition 4(2) and Definition 2.1.5. (4) follows from (2) and Lemma 3.3.1. (5) follows from Proposition 4(5). (6) follows from Lemma 3.5.7. Q.E.D.

**Lemma 5.1.4.** Suppose that  $j_0 \neq \emptyset$ . Then there is a neighborhood of  $\partial V_{l+1}^l$  in  $\partial_+ \mathcal{V}_-$ , say  $\partial V_{l+1}^l \times I$ , satisfying the following conditions:

- (1)  $\partial V_{l+1}^l \times I \cap F^l = c_{l+1}^l \times I$ .
- (2) If  $j, \gamma < j_0$ , then  $d_j^l$  and  $e_\gamma^l \times I$  are disjoint from  $c_{l+1}^l \times I$ .
- (3)  $\partial d_j^l, (\partial e_\gamma^l) \times I$  are disjoint from  $c_{l+1}^l \times I$  for each  $j \notin m(l)$  and  $\gamma \in m(l)$ .

(4)  $d_{j_0}^l$  intersects  $c_{l+1}^l \times I$  in an arc  $a \subset \text{int}d_{0,0}$ , and  $d_{j_0}^l \times [-2.5, 2.5]$  intersects  $c_{l+1}^l \times I$  in  $a \times [-2.5, 2.5]$ . See Figure 19.

(5) For  $i \geq l + 2$ ,  $c_i^l$  is disjoint from  $c_{l+1}^l \times I$ , if  $s(v_i) = -$ , then  $V_i^l$  is disjoint from  $\partial V_{l+1}^l \times I$ .

(6) For each  $j \notin m(l)$  with  $s(w_j) = -$ ,  $W_j^l - d_j^l \cup_{r \in I(w_j, l)} d_r^l$  is disjoint from  $\partial V_{l+1}^l \times I$ .

**Proof.** (1) follows from Lemma 5.1.1(1). (2) follows from Lemma 5.1.1(3) and (4). (3) follows from Proposition 4(5). (4) follows from Lemma 5.1.1(2). (5) and (6) follow from Proposition 6. Q.E.D.

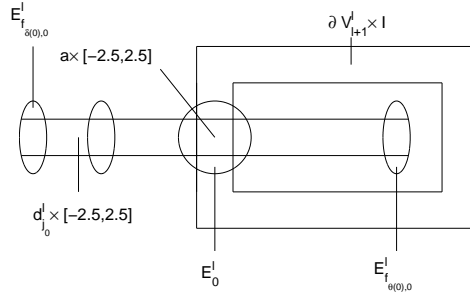


Figure 19

**Lemma 5.1.5.** If  $L(c_{l+1}^l) = \emptyset$ , then Propositions 4-6 hold for the case:  $k = l + 1$  and  $s(v_{l+1}) = -$ .

**Proof.** By Corollary 3.2.5,  $d_j^l, e_\gamma^l \times I$  are disjoint from  $c_{l+1}^l \times I$  where  $c_{l+1}^l \times I$  is a regular neighborhood of  $c_{l+1}^l$  in  $E_0^l$ . Let  $F^{l+1} = F^l - c_{l+1}^l \times (-1, 1)$ . We denote by  $d_j^{l+1}$  the arc  $d_j^l$  for  $j \in \{1, \dots, n\} - m(l)$ ,  $e_\gamma^{l+1} \times I$  the disk  $e_\gamma^l \times I$  for  $\gamma \in m(l)$ ,  $E_f^{l+1}$  the disk  $E_f^l$  for  $1 \leq f \leq l$ . In particular, we denote by  $E_0^{l+1}, E_{l+1}^{l+1}$  the two components of  $E_0^l - c_{l+1}^l \times (-1, 1)$ . By Proposition 4,  $c_i^l \cap c_{l+1}^l = \emptyset$  for  $i \geq l + 2$ . Now we denote by  $c_i^{l+1}$  the arc  $c_i^l$ . Let  $m(l+1) = m(l)$ . For  $i \geq l + 2$  with  $s(v_i) = -$  and  $j \in \{1, \dots, n\} - m(l)$ , we denote by  $V_i^{l+1}$  the disk  $V_i^l$ ,  $W_j^{l+1}$  the disk  $W_j^l$ . By the argument in Section 4.2, the lemma holds. Q.E.D.

## 5.2 The constructions of $c_i^{l+1}, d'_j, e'_\gamma \times I, F^{l+1}$ (I).

By Lemma 5.1.5, in the following argument, we may assume that  $L(c_{l+1}^l) \neq \emptyset$ . Hence  $j_0 = m^{l+1} \neq \emptyset$ . In Sections 5.2 and 5.3, we shall construct  $c_i^{l+1}, d'_j, e'_\gamma \times I, F^{l+1}$  from  $d_j^l, c_i^l, e_\gamma^l \times I, F^l$ , where  $c_i^{l+1}, F^{l+1}$  are just  $c_i^{l+1}, F^{l+1}$  in Propositions 4-6, but  $d'_j, e'_\gamma \times I$  are not  $d_j^{l+1}, e_\gamma^{l+1} \times I$  in Propositions 4-6. In this section, we first assume that  $j_0 \notin I(w_\gamma, l)$  for each  $\gamma \in m(l)$  with  $s(w_\gamma) = -$ .

Let  $\partial V_{l+1}^l \times I$  be a neighborhood of  $\partial V_{l+1}^l$  in  $\partial_+ \mathcal{V}_-$  satisfying Lemma 5.1.4, and  $d_{j_0}^l \times [-2.5, 2.5]$  be a neighborhood of  $d_{j_0}^l$  in  $F^l$  satisfying Lemma 5.1.3.

**Definition 5.2.1.** We denote by  $E_0, E_{l+1}$  the two components of  $E_0^l - c_{l+1}^l \times (-1, 1)$ .

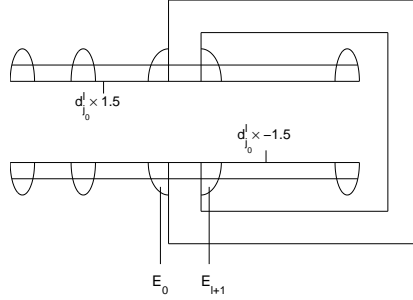


Figure 20

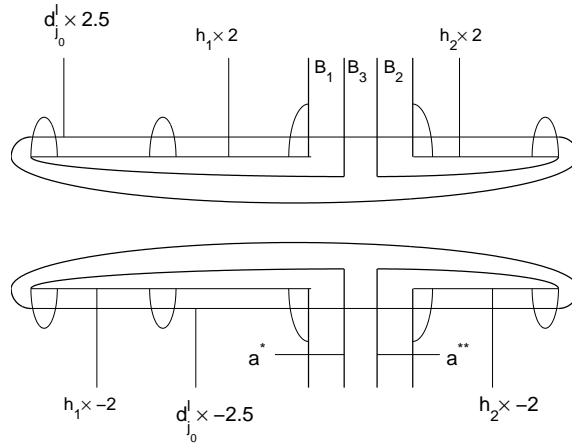


Figure 21

**Definition 5.2.2.** (1) Let  $F^* = F^l \cup \partial V_{l+1}^l \times I - d_{j_0}^l \times (-1.5, 1.5)$ . See Figure 20.

(2) Let  $F^{l+1}$  be the surface obtained by identifying  $\partial_1 d_{j_0}^l \times [1.5, 2]$  to  $\partial_1 d_{j_0}^l \times [2, 2.5]$  and  $\partial_1 d_{j_0}^l \times [-2, -1.5]$  to  $\partial_1 d_{j_0}^l \times [-2.5, -2]$  in  $\partial_+ \mathcal{V}_- - F^{l+1}$ . See Figure 21.

Let  $b = \partial V_{l+1}^l - a \times (-2, 2)$ . Then  $b \times I = (\partial V_{l+1}^l - a \times (-2, 2)) \times I$ . Now by Lemma 5.1.4(4),  $B = (d_{j_0}^l \times [-2, -1.5] \cup [1.5, 2]) \cup b \times I$  is a disk. We denote by  $h_1, h_2$  the two components of  $d_{j_0}^l - \text{inta}$  where  $a$  is as in Lemma 5.1.4. Now there is an arc  $a^*$  in  $B = (d_{j_0}^l \times [-2, -1.8] \cup [1.8, 2]) \cup b \times I$  connecting  $\partial_1 d_{j_0}^l \times \{2\}$  to  $\partial_1 d_{j_0}^l \times \{-2\}$ . Furthermore,  $a^*$  intersects  $b \times I$  in  $b \times \{-1/2\}$ . Similarly, there is an arc  $a^{**}$  in  $(d_{j_0}^l \times [-2, -1.8] \cup [1.8, 2]) \cup b \times I$  connecting  $\partial_2 d_{j_0}^l \times \{2\}$  to  $\partial_2 d_{j_0}^l \times \{-2\}$ . Furthermore,  $a^{**}$  intersects  $b \times I$  in  $b \times \{1/2\}$ . Now  $a^* \cup a^{**}$  separates  $B = (d_{j_0}^l \times [-2, -1.5] \cup [1.5, 2]) \cup b \times I$  into three disks  $B_1, B_2, B_3$  as in Figure 21. Without loss of generality, we may assume that  $h_1 \times \{-2, 2\} \subset B_1$  and  $h_2 \times \{-2, 2\} \subset B_2$  as in Figure 21. Note that  $B_1 \cup B_2 \subset (d_{j_0}^l \times [-2, -1.8] \cup [1.8, 2]) \cup b \times I$ .

Now there is a homeomorphism  $H_i$  from  $h_i \times [-2, 2] \subset d_{j_0}^l \times [-2, 2]$  to  $B_i$  such that  $H_i$  is an identifying map on  $h_i \times \{-2, 2\}$  for  $i = 1, 2$ . Let  $H = H_1 \cup H_2$ .

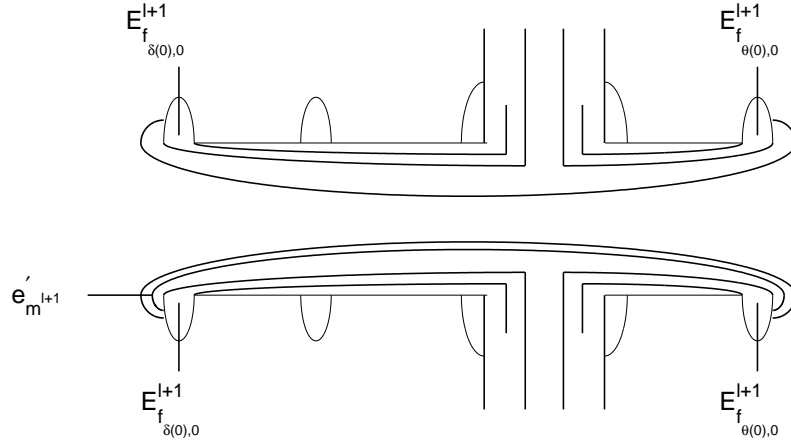


Figure 22

By Lemma 3.3.1 and Remark 3.3.2,  $d_{j_0}^l = \cup_{i=\delta(0)}^{\theta(0)} d_{i,0} \cup_{i=\delta(0)}^{\theta(0)} e_{i,0}$ .

**Definition 5.2.3.** (1) For each  $\gamma \in m(l)$ , let  $e'_\gamma = e_\gamma^l$ ,  $e'_\gamma \times I = (e_\gamma^l \times I - e_\gamma^l \times I \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(e_\gamma^l \times I \cap (h_1 \cup h_2) \times [-2, 2])$ .

(2) For  $j \notin m(l+1)$ , let  $d'_j = d_j^l$ .

(3) For  $i \geq l+2$ , let  $c_i^{l+1} = (c_i^l - c_i^l \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(c_i^l \cap (h_1 \cup h_2) \times [-2, 2])$ .

(4) For  $1 \leq f \leq l$ , let  $E_f^{l+1} = (E_f^l - E_f^l \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(E_f^l \cap (h_1 \cup h_2) \times [-2, 2])$ , let  $E_0^{l+1} = (E_0 - E_0 \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(E_0 \cap (h_1 \cup h_2) \times [-2, 2])$ , let  $E_{l+1}^{l+1} = (E_{l+1} - E_{l+1} \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(E_{l+1} \cap (h_1 \cup h_2) \times [-2, 2])$ .

(5) Let  $e_{m^{l+1}}^{l+1} = d_{j_0}^l \times \{-1.6\}$ , and  $e_{m^{l+1}}^{l+1} \times I = B_3$  with  $e_{m^{l+1}}^{l+1} \times [-1, 0] = d_{j_0}^l \times [-1.5, -1.6]$ .

See Figure 22.

### 5.3 The constructions of $c_i^{l+1}, d_j', e_\gamma' \times I, F^{l+1}$ (II).

In this section, we shall construct  $c_i^{l+1}, d_j', e_\gamma' \times I, F^{l+1}$  from  $d_j^l, c_i^l, e_\gamma^l \times I, F^l$  for the case:  $j_0 \in I(w_{\gamma_0}, l)$  for some  $\gamma_0 \in m(l)$  with  $s(w_{\gamma_0}) = -$ .

Let  $\partial V_{l+1}^l \times I$  be a neighborhood of  $\partial V_{l+1}^l$  in  $\partial_+ \mathcal{V}_-$  satisfying Lemma 5.1.4, and  $d_{j_0}^l \times [-2, 2]$  be a neighborhood of  $d_{j_0}^l$  in  $F^l$  satisfying Lemma 5.1.2.

Recalling the disks  $E_0, E_{l+1}, B_1, B_2, B_3$  and the arcs  $h_1, h_2, a^*, a^{**}$  in Section 5.2.

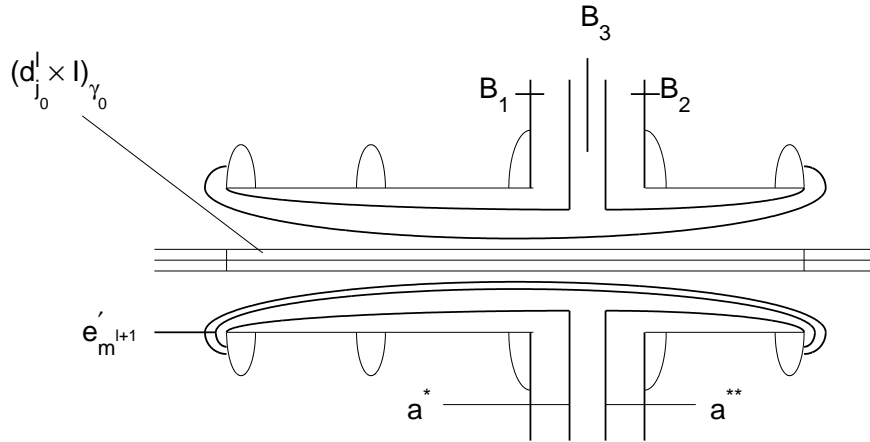


Figure 23

**Definition 5.3.1.** (1) Let  $F^* = (F^l \cup \partial V_{l+1}^l \times I - d_{j_0}^l \times (-1.5, 1.5)) \cup (d_{j_0}^l \times I)_{\gamma_0}$ .

(2) Let  $F^{l+1}$  be the surface obtained by identifying  $\partial_1 d_{j_0}^l \times [1.5, 2]$  to  $\partial_1 d_{j_0}^l \times [2, 2.5]$  and  $\partial_1 d_{j_0}^l \times [-2, -1.5]$  to  $\partial_1 d_{j_0}^l \times [-2.5, -2]$  in  $\partial_+ \mathcal{V}_- - F^{l+1}$ . See Figure 23.

**Definition 5.3.2.** (1) For each  $\gamma \in m(l)$  with  $\gamma < m^{l+1} = j_0$ , let  $e_\gamma' = e_\gamma^l$ ,  $e_\gamma' \times I = (e_\gamma^l \times I - e_\gamma^l \times I \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(e_\gamma^l \times I \cap (h_1 \cup h_2) \times [-2, 2])$ .

(2) For each  $\gamma \in m(l)$  with  $\gamma > m^{l+1} = j_0$ , let  $e_\gamma' = e_\gamma^l$ ,  $e_\gamma' \times I = e_\gamma^l \times I$ .

- (3) Let  $e_{m^{l+1}}^{l+1} = d_{j_0}^l \times \{-1.6\}$ , and  $e_{m^{l+1}}^{l+1} \times I = B_3$  with  $e_{m^{l+1}}^{l+1} \times [-1, 0] = d_{j_0}^l \times [-1.5, -1.6]$ .

See Figure 24.

**Definition 5.3.3.** (1) For  $j \notin m(l+1)$ , let  $d'_j = d_j^l$ .  
(2) For  $i \geq l+2$ , let  $c_i^{l+1} = (c_i^l - c_i^l \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(c_i^l \cap (h_1 \cup h_2) \times [-2, 2])$ .  
(3) For  $1 \leq f \leq l$ , let  $E_f^{l+1} = (E_f^l - E_f^l \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(E_f^l \cap (h_1 \cup h_2) \times [-2, 2])$ , let  $E_0^{l+1} = (E_0 - E_0 \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(E_0 \cap (h_1 \cup h_2) \times [-2, 2])$ , let  $E_{l+1}^{l+1} = (E_{l+1} - E_{l+1} \cap (h_1 \cup h_2) \times [-2, 2]) \cup H(E_{l+1} \cap (h_1 \cup h_2) \times [-2, 2])$ . See Figures 23 and 24.

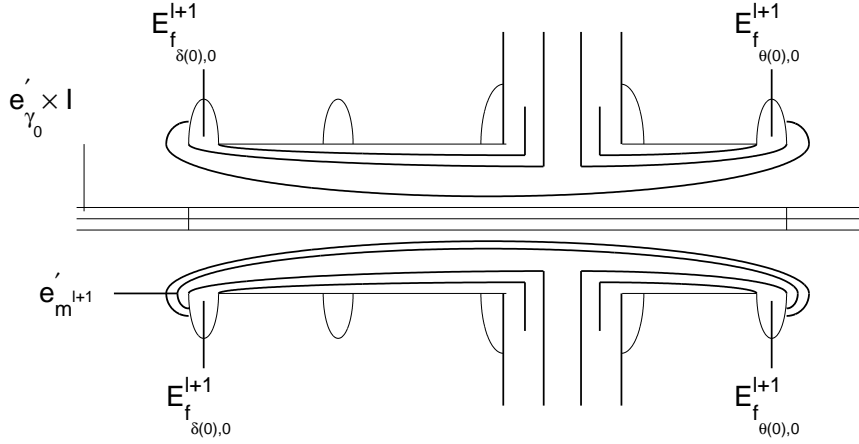


Figure 24

## 5.4 Properties of $c_i^{l+1}, d'_j, e'_\gamma \times I, F^{l+1}$

In this section, we shall introduce simple properties of  $c_i^{l+1}, d'_j, e'_\gamma \times I, F^{l+1}$ .

**Lemma 5.4.1.**  $(d_{j_0}^l \times [-2, -1.5] \cup [1.5, 2]) \cup b \times I \subset E_0^{l+1} \cup E_{l+1}^{l+1} \cup_{i=\delta(0)}^{\theta(0)} E_{f_{i,0}}^{l+1} \cup_{i=\delta(0)}^{\theta(0)} e'_{\gamma_{i,0}} \times I \cup e'_{m^{l+1}} \times I$ .

**Proof.** By Lemma 3.3.1,  $d_{j_0}^l = \cup_{i=\delta(0)}^{\theta(0)} d_{i,0} \cup_{i=\delta(0)}^{\theta(0)} e_{i,0}$ , where  $d_{i,0}$  is an properly embedded arc in  $E_{f_{i,0}}^l$ ,  $e_{i,0}$  is a core of  $e_{\gamma_{i,0}}^l \times (0, 1)$  for  $i \neq 0$ , and  $d_{0,0}$  is an arc in  $E_0^l$ . By Lemma 5.1.4,  $a \subset \text{int} d_{0,0}$ . We denote by  $h^*, h^{**}$  the two components of  $d_{0,0} - \text{inta}$ . We may assume that  $h^* \subset E_0$  and  $h^{**} \subset E_{l+1}$ . Then  $h_1 \times [-2, 2] = (\cup_{i=\delta(0)}^{-1} d_{i,0} \cup h^* \cup_{i=\delta(0)}^{-1} e_{i,0}) \times [-2, 2]$  and  $h_2 \times [-2, 2] = (h^{**} \cup_{i=1}^{\theta(0)} d_{i,0} \cup_{i=1}^{\theta(0)} e_{i,0}) \times [-2, 2]$ . Note that  $(d_{j_0}^l \times [-2, -1.5] \cup [1.5, 2]) \cup b \times I = B_1 \cup B_2 \cup B_3$ . Now by Definitions 5.2.3, 5.3.2 and 5.3.3, the lemma holds. Q.E.D.

**Lemma 5.4.2.**  $\{d'_j \mid j \in \{1, 2, \dots, n\} - m(l+1)\}$  is a set of pairwise disjoint arcs in  $F^{l+1}$ . Furthermore,  $d'_j$  is properly embedded in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e'_\gamma \times I$ .

**Proof.** Now there are two cases:

Case 1.  $j_0 \in I(w_{\gamma_0}, l)$  for some  $\gamma_0 \in m(l)$  with  $s(w_{\gamma_0}) = -$ .

By Lemma 5.1.2,  $d'_j = d_j^l$  is disjoint from  $d_{j_0}^l \times [-1.5, 1.5]$ . By Proposition 4(2),  $d_j^l$  is properly embedded in  $\cup_f E_f^l \cup_{\gamma < j} e'_\gamma \times I$ . If  $j < j_0$ , then, by Lemma 5.1.4,  $d_j^l$  is disjoint from  $\partial V_{l+1}^l \times I$ . Hence  $d_j^l$  is also properly embedded in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e'_\gamma \times I$ . If  $j > j_0 = m^{l+1}$ , then  $j > \gamma_{i,0}$  for each  $\delta(0) \leq i \leq \theta(0)$ . By Definitions 5.3.2, 5.3.3 and Lemma 5.4.1,  $d_j^l \cap \partial V_{l+1}^l \times I = d_j^l \cap c_{l+1}^l \times I \subset E_0^{l+1} \cup E_{l+1}^{l+1} \cup_{i=\delta(0)}^{\theta(0)} E_{f_{i,0}}^{l+1} \cup_{i=\delta(0)}^{\theta(0)} e'_{\gamma_{i,0}} \times I \cup e'_{m^{l+1}} \times I \cup (d_{j_0}^l \times I)_{\gamma_0}$ . By Lemma 5.1.2, if  $d_j^l \cap (d_{j_0}^l \times I)_{\gamma_0} \neq \emptyset$ , then  $j > \gamma_0$ . Hence the lemma holds.

Case 2.  $s(w_{\gamma_0}) = +$  or  $\gamma_0 = \emptyset$ .

By Lemma 5.1.3 and the argument in Case 1, the lemma holds. Q.E.D.

**Lemma 5.4.3.** (1)  $E_f^{l+1}$  is a disk in  $F^{l+1}$ .

(2)  $\{c_i^{l+1} \mid i \geq l+2\}$  is a set of pairwise disjoint arcs properly embedded in  $F^{l+1}$ .

Furthermore,  $c_i^{l+1}$  lies in one of  $E_f^{l+1}$  for some  $f$ .

(3)  $\partial c_i^{l+1} \cap e'_\gamma \times I = \emptyset$  for each  $\gamma \in m(l+1)$ , and  $\partial c_i^{l+1} \cap d'_j = \emptyset$  for each  $j \in \{1, 2, \dots, n\} - m(l+1)$ .

(4)  $c_i^{l+1}$  is obtained by doing band sums with copies of  $\partial V_{l+1}^l$  to  $c_i^l$ .

**Proof.** (1) For each  $1 \leq f \leq l$ , by Lemma 3.1.6, each component of  $d_{j_0}^l \cap E_f^l$  is a properly embedded arc  $c$  in  $E_f^l$ , and each component of  $d_{j_0}^l \times [-2, 2] \cap E_f^l$  is  $c \times [-2, 2]$  in  $E_f^l$ . Since  $E_f^l \cap E_0^l = \emptyset$ ,  $E_f^{l+1}$  is a disk in  $F^{l+1}$ .

Since each component of  $d_{j_0}^l \cap E_0^l$  is a properly embedded arc  $c$  in  $E_0^l$ . By Lemma 5.1.4(4), each component of  $d_{j_0}^l \times [-2, 2] \cap E_f$  is  $(c \cap E_f) \times [-2, 2]$  in  $E_f^l$ , where  $c \cap E_f$  is a properly embedded arc in  $E_f$  for  $f = 0, l+1$ . Hence  $E_0^{l+1}, E_{l+1}^{l+1}$  are two disks in  $F^{l+1}$ .

(2) Let  $i \geq l+2$ . By Proposition 4(4) and Lemma 5.1.4(5), either  $c_i^l$  lies in  $E_f^l$  for some  $1 \leq f \leq l$  or  $c_i^l$  lies in  $E_f$  for  $f = 0, l+1$ . We may assume that  $c_i^l \subset E_f^l$ . By Lemma 3.1.6, each component of  $c_i^l \cap d_{j_0}^l$  is a point  $p$ . Furthermore,  $p = c_i^l \cap c$  where  $c$  is one component of  $d_{j_0}^l \cap E_f^l$ . By Lemma 5.1.2(8) and Lemma 5.1.3(5), each component of

$d_{j_0}^l \times [-2, 2] \cap c_i^l = p \times [-2, 2] \subset \text{int} c_i^l$ . By the construction,  $c_i^{l+1}$  is a properly embedded arc in  $F^{l+1}$  which lies in  $E_f^{l+1}$ .

(3) By Lemma 5.1.2(8) and Lemma 5.1.3(5),  $\partial c_i^l$  is disjoint from  $d_{j_0}^l \times [-2.5, 2.5]$ . By Proposition 4(5) and the construction, (3) holds.

(4) Since  $\partial V_{l+1}^l \times I \cup d_{j_0}^l \times [-2, 2] - d_{j_0}^l \times (-1.5, 1.5)$  is a disk. By (2), each component of  $c_i^l \cap d_{j_0}^l \times [-2, 2]$  is an arc. Hence the homeomorphism  $H$  means band sums. Q.E.D.

**Lemma 5.4.4.** For each  $\gamma \in m(l+1)$ ,  $e_\gamma' \times I$  is a disk in  $F^{l+1}$  such that

- (1)  $(\partial e_\gamma') \times I \subset \cup_f \partial E_f^{l+1}$  for  $\gamma \in m(l)$ ,
- (2)  $(\partial e_\gamma') \times I \cap (\partial e_\lambda') \times I = \emptyset$  for  $\gamma \neq \lambda$ .
- (3)  $\partial_1 e_{m^{l+1}}' \subset E_{f_{\delta(0),0}}^{l+1}$  and  $\partial_2 e_{m^{l+1}}' \subset E_{f_{\theta(0),0}}^{l+1}$ .
- (4)  $e_\gamma' \times I = e_\gamma^l \times I$  for  $\gamma \neq m^{l+1} > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

**Proof.** There are two cases:

Case 1.  $s(w_{\gamma_0}) = -$ .

Suppose that  $\gamma \neq m^{l+1} > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . If  $\gamma \geq \gamma_0$ , then  $\gamma > j_0$ . By Definition 5.3.2,  $e_\gamma' \times I = e_\gamma^l \times I$ . By Lemma 5.1.2(4),  $e_\gamma^l \times I$  is disjoint from  $d_{j_0}^l \times [-2.5, -1.5] \cup [1.5, 2.5]$ . Hence  $e_\gamma' \times I \subset F^{l+1}$ . If  $\gamma < \gamma_0$ , then, by Lemma 5.1.2(3),  $e_\gamma^l \times I$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . By Definition 5.3.2,  $e_\gamma' \times I = e_\gamma^l \times I \subset F^{l+1}$ .

Suppose that  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . By Lemma 5.1.2, each component of  $d_{j_0}^l \times [-2, 2] \cap e_\gamma^l \times I$  is  $c \times [-2, 2] \subset e_\gamma^l \times (0, 1)$ . By Lemma 5.1.4,  $e_\gamma^l \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Definition 5.3.2,  $e_\gamma' \times I$  is a disk in  $F^{l+1}$ .

It is easy to see that  $e_{m^{l+1}}' \times I = B_3$  is a disk in  $F^{l+1}$ .

By Proposition 4(1) and Definition 2.1.4,  $(\partial e_\gamma^l) \times I \subset \cup_f \partial E_f^l$ ,  $(\partial e_\gamma^l) \times I \cap (\partial e_\lambda^l) \times I = \emptyset$  for  $\gamma, \lambda \in m(l)$ . Hence (1) follows from the construction, and  $(\partial e_\gamma') \times I \cap (\partial e_\lambda') \times I = \emptyset$  for  $\gamma \neq \lambda \in m(l)$ . By Proposition 4(1) and Lemma 5.1.2,  $\partial d_{j_0}^l \times [-2, 2]$  is disjoint from  $e_\gamma^l \times I$  for  $\gamma \in m(l)$  even if  $\gamma = \gamma_0$ . By Lemma 5.3.2,  $(\partial e_{m^{l+1}}') \times I = H(\partial d_{j_0}^l \times [-2, 2])$ . Hence  $(\partial e_{m^{l+1}}') \times I \cap (\partial e_\gamma') \times I = \emptyset$  for  $\gamma \in m(l)$ . Thus (2) holds.

By Lemma 3.3.1,  $d_{j_0}^l = \cup_{i=\delta(0)}^{\theta(0)} d_{i,0} \cup_{i=\delta(0)}^{\theta(0)} e_{i,0}$ . Hence  $\partial_1 d_{j_0}^l \subset E_{f_{\delta(0),0}}^l$  and  $\partial_2 d_{j_0}^l \subset E_{f_{\theta(0),0}}^l$ . By Definitions 5.3.1, 5.3.2 and 5.3.3, (3) holds. See Figure 24.



Case 2.  $s(w_{\gamma_0}) = +$  or  $\gamma_0 = \emptyset$ .

By Lemma 5.1.3 and the argument in Case 1, the lemma holds for this case. Q.E.D.

## 5.5 $F^{l+1}$ is a surface generated by $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e'_\gamma$

**Lemma 5.5.1.**  $F^{l+1} = \cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e'_\gamma \times I$ .

**Proof.** By Lemma 5.4.1,  $(d_{j_0}^l \times ([ -2, -1.5] \cup [1.5, 2])) \cup b \times I \subset \cup_f E_f^{l+1} \cup_{i=\delta(0)}^{\theta(0)} e'_{\gamma_{i,0}} \times I \cup e'_{m^{l+1}} \times I$ . By Proposition 4(1), the lemma holds. Q.E.D.

**Lemma 5.5.2.** If  $s(w_\lambda) = +$ , then  $inte'_\lambda \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I$ .

**Proof.** By Proposition 4(1), if  $\lambda \in m(l)$  with  $s(w_\lambda) = +$ , then  $inte'_\lambda \times I$  is disjoint from  $\cup_f E_f^l \cup_{\gamma < \lambda} e'_\gamma \times I$ . There are two cases:

Case 1.  $s(w_{\gamma_0}) = -$ .

There are three sub-cases:

Case 1.1.  $\lambda \neq m^{l+1} > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

By Lemma 5.4.3,  $e'_\lambda \times I = e'_\lambda \times I$ . By Proposition 4(1),  $inte'_\lambda \times I$  is disjoint from  $\cup_f E_f^l \cup_{\gamma < \lambda} e'_\gamma \times I$ . By Lemma 5.1.4(1), Proposition 4(5),  $inte'_\lambda \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Lemma 5.1.2(3) and (4),  $d_{j_0}^l \times [-2 - 1.5] \cup [1.5, 2]$  is disjoint from  $e'_\lambda \times I$ . By Definitions 5.3.2 and 5.3.3,  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I \subset \cup_f E_f^l \cup_{\gamma < \lambda} e'_\gamma \times I \cup \partial V_{l+1}^l \times I \cup d_{j_0}^l \times [-2 - 1.5] \cup [1.5, 2]$ . Hence  $inte'_\lambda \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I$ .

Case 1.2.  $\lambda = m^{l+1}$ . Now if  $\gamma < \lambda$ , then, by Lemma 5.1.4(2),  $e'_\gamma \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Definitions 5.3.2 and 5.3.3,  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I \subset (\cup_f E_f^l \cup_{\gamma < \lambda} e'_\gamma \times I - d_{j_0}^l \times [-2, 2]) \cup B_1 \cup B_2$ ,  $e'_{m^{l+1}} \times I = B_3$  where  $B_i$  is as in Section 5.2. Hence the lemma holds.

Case 1.3.  $\lambda \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . Now if  $\gamma \leq \lambda$ , then  $\gamma \leq \lambda < j_0 = m^{l+1}$ . By Lemma 5.1.4,  $e'_\gamma \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Proposition 4(1),  $inte'_\lambda \times I$  is disjoint from  $\cup_f E_f^l \cup_{\gamma < \lambda} e'_\gamma \times I$ . By Lemma 5.1.2, each component of  $d_{j_0}^l \times [-2, 2] \cap e'_\lambda \times I$  is  $c \times [-2, 2] \subset e'_\lambda \times (0, 1)$  where  $c$  is a core of  $e'_\lambda \times (0, 1)$ . Hence  $intc \times [-2, 2]$  is disjoint from  $\cup_f E_f^l \cup_{\gamma < \lambda} e'_\gamma \times I$ . By Definitions 5.3.2 and 5.3.3, the lemma holds.

Case 2.  $s(w_{\gamma_0}) = +$  or  $\gamma_0 = \emptyset$ .

By Lemma 5.1.3, Definition 5.2.3 and the argument in Case 1, the lemma holds. Q.E.D.

**Lemma 5.5.3.** If  $s(w_\lambda) = -$ , then  $\text{inte}'_\lambda \times I \cup (\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I) = \cup_{r \in I(w_\lambda, l+1)} (d'_r \times I)_\lambda$ .

**Proof.** Suppose that  $\lambda \in m(l+1)$  and  $s(w_\lambda) = -$ . By Lemma 5.1.4(4),  $\lambda \neq m^{l+1}$ . By Proposition 4(1),  $\text{inte}^l_\lambda \times I \cup (\cup_f E_f^l \cup_{\gamma < \lambda} e^l_\gamma \times I) = \cup_{r \in I(w_\lambda, l)} (d^l_r \times I)_\lambda$ . Now there are two cases:

Case 1.  $s(w_{\gamma_0}) = -$ . There are four sub-cases:

Case 1.1.  $\lambda \neq \gamma_0 > m^{l+1}$ .

In this case,  $\{\gamma < \lambda \in m(l+1)\} = \{\gamma < \lambda \in m(l)\} \cup \{m^{l+1}\}$ .

By Definition 2.3.1,  $I(w_\lambda, l+1) = I(w_\lambda, l) - \{m^{l+1}\}$ . Since  $m^{l+1} = j_0 \in I(w_{\gamma_0}, l)$ , by Lemma 2.2.5,  $I(w_\lambda, l+1) = I(w_\lambda, l)$ . By Lemma 5.4.4,  $e'_\gamma \times I = e^l_\gamma \times I$ . By Lemmas 5.1.2,  $d^l_{j_0} \times [-2, -1] \cup [1, 2]$  is disjoint from  $e^l_\lambda \times I$ . By Proposition 4(1),  $e^l_\lambda \times I \cap (\cup_f E_f^l \cup_{\gamma < \lambda} e^l_\gamma \times I - d^l_{j_0} \times [-2, -1] \cup (1, 2]) = \cup_{r \in I(w_\lambda, l)} (d^l_r \times I)_\lambda$ . By Proposition 4(5) and Lemma 5.1.4(1),  $e^l_\lambda \times I$  is disjoint from  $(\partial V_{l+1}^l - c_{l+1}^l) \times I$ . By Lemma 5.4.1, the lemma holds.

Case 1.2.  $\lambda = \gamma_0$ .

In this case,  $\{\gamma < \lambda \in m(l+1)\} = \{\gamma < \lambda \in m(l)\} \cup \{m^{l+1}\}$ .

By Definition 2.3.1,  $I(w_\lambda, l+1) = I(w_\lambda, l) - \{m^{l+1}\}$ .

Now by Lemma 5.1.2(2), (3) and Definitions 5.3.2, 5.3.3,  $(d^l_{j_0} \times I)_{\gamma_0}$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \gamma_0} e'_\gamma \times I$ . By the argument in Case 1.1, the lemma holds.

Case 1.3.  $m^{l+1} > \lambda > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

In this case,  $\{\gamma < \lambda \in m(l+1)\} = \{\gamma < \lambda \in m(l)\}$ .

By Lemma 5.1.4 and Lemma 5.4.4,  $e'_\lambda \times I = e^l_\lambda \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Lemma 5.1.2(3),  $e^l_\lambda \times I$  is disjoint from  $d^l_{j_0} \times [-2, 2]$ . By Definition 5.3.2 and 5.3.3,  $e^l_\lambda \times I \cap (\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I) = e^l_\lambda \times I \cap (\cup_f E_f^l \cup_{\gamma < \lambda} e^l_\gamma \times I)$ . Hence the lemma holds.

Case 1.4.  $\lambda \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

In this case,  $\{\gamma < \lambda \in m(l+1)\} = \{\gamma < \lambda \in m(l)\}$ .

Now we denote by  $P_\lambda$  the surface  $\cup_f E_f^l \cup_{\gamma < \lambda} e^l_\gamma \times I$ .

Now  $e'_\lambda \times I \cap (\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I) = ((e^l_\lambda \times I - e^l_\lambda \times I \cap d^l_{j_0} \times [-2, 2]) \cap (P_\lambda - P_\lambda \cap d^l_{j_0} \times [-2, 2])) \cup (H(e^l_\lambda \times I \cap d^l_{j_0} \times [-2, 2]) \cap H(P_\lambda \cap d^l_{j_0} \times [-2, 2]))$ . Hence the lemma holds.

Case 2.  $s(w_{\gamma_0}) = +$  or  $\gamma_0 = \emptyset$ .

By Lemma 5.1.3 and Definition 5.2.3, and the argument in Case 1, the lemma holds.  
Q.E.D.

**Lemma 5.5.4.** If  $\gamma < \lambda \in m(l+1)$ , then each component of  $e'_\lambda \times I \cap e'_\gamma \times I$  is  $(c \times I)_\lambda \subset e'_\gamma \times (0, 1)$  where  $c \subset \text{inte}'_\lambda$  is a core of  $e'_\gamma \times (0, 1)$ .

**Proof.** By Proposition 4(1) and Definition 2.1.4, if  $\gamma < \lambda \in m(l)$ , then each component of  $e'_\lambda \times I \cap e'_\gamma \times I$  is  $(c \times I)_\lambda \subset e'_\gamma \times (0, 1)$  where  $c \subset \text{inte}'_\lambda$  is a core of  $e'_\gamma \times (0, 1)$ .

Without loss of generality, we may assume that  $s(w_{\gamma_0}) = -$ . Now there are six cases:

Case 1.  $\lambda, \gamma \in m(l)$  and  $\lambda > \gamma > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

By Lemma 5.4.3,  $e'_\lambda \times I = e'_\lambda \times I$  and  $e'_\gamma \times I = e'_\gamma \times I$ . Hence the lemma holds.

Case 2.  $\lambda = m^{l+1} > \gamma$ .

Since  $s(w_{m^{l+1}}) = +$ , by Lemma 5.4.4(2) and Lemma 5.5.2,  $e'_{m^{l+1}} \times I$  is disjoint from  $e'_\gamma \times I$ .

Case 3.  $\lambda > \gamma = m^{l+1}$ .

By Lemma 3.2.4, and Lemma 5.1.4(1), each component of  $e'_\lambda \times I \cap e'_{l+1} \times I = e'_\lambda \times I \cap \partial V_{l+1}^l \times I$  is  $(c \times I)_\lambda$  where  $c \subset \text{inte}'_\lambda$ . Furthermore, by Lemma 5.1.2(3) and (4),  $(c \times I)_\lambda$  lies either in  $F^l - d_{j_0}^l \times [-2, 2]$  or in  $(d_{j_0}^l \times I)_{\gamma_0}$ . If  $(c \times I)_\lambda$  lies in  $(d_{j_0}^l \times I)_{\gamma_0}$ . Then  $(c \times I)_\lambda$  is disjoint from  $e'_{m^{l+1}} \times I$ . If  $(c \times I)_\lambda$  lies in  $F^l - d_{j_0}^l \times [-2, 2]$ , then, by Definitions 5.3.3,  $(c \times I)_\lambda$  intersects  $e'_{m^{l+1}} \times I$  in  $(c_* \times I)_\lambda \subset e'_{m^{l+1}} \times (0, 1)$ , where  $c_* \subset \text{inte}'_\lambda$  is a core of  $e'_{m^{l+1}} \times (0, 1)$ . Hence the lemma holds.

Case 4.  $\gamma_0 > \lambda > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\} \geq \gamma$ .

By Lemma 5.4.4(4),  $e'_\lambda = e'_\lambda \times I$ . By Lemma 5.1.2(3),  $e'_\lambda \times I$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . By Lemma 5.1.4,  $e'_\lambda \times I \cap \partial V_{l+1}^l \times I = e'_\lambda \times I \cap e'_{l+1} \times I$ . By Lemma 3.2.4, each component of  $e'_\lambda \times I \cap \partial V_{l+1}^l \times I$  is  $(c \times I)_\lambda$  where  $c \subset \text{inte}'_\lambda$ . By Lemma 5.1.4,  $e'_\gamma \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Definition 5.3.2,  $e'_\lambda \times I \cap e'_\gamma \times I = S_1 \cup S_2$ , where  $S_1 = e'_\lambda \times I \cap e'_\gamma \times I$  and  $S_2 = (e'_\lambda \times I \cap \partial V_{l+1}^l \times I) \cap H(e'_\gamma \cap (h_1 \cup h_2) \times [-2, 2])$ . Hence the lemma holds.

Case 5.  $\lambda \geq \gamma_0 > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\} \geq \gamma$ .

Now by Lemma 5.1.2 and Lemma 5.4.4,  $e'_\lambda \times I = e'_\lambda \times I$  is disjoint from  $d_{j_0}^l \times [-2, -1) \cup (1, 2]$  even if  $\lambda = \gamma_0$ . By Definition 5.3.2,  $e'_\lambda \times I \cap e'_\gamma \times I = S_1 \cup S_2$ , where  $S_1 = (e'_\lambda \times I -$

$e_\lambda^l \times I \cap d_{j_0}^l \times [-1, 1] \cap e_\gamma^l \times I$  and  $S_2 = (e_\lambda^l \times I \cap \partial V_{l+1} \times I) \cap H(e_\gamma^l \times I \cap (h_1 \cup h_2) \times I)$ .

By Lemma 3.3.1,  $d_{j_0}^l = \cup_{i=\delta(0)}^{\theta(0)} d_{i,0} \cup_{i=\delta(0)}^{\theta(0)} e_{i,0}$ . Since  $\gamma < j_0$ , by Lemma 5.1.2(6),  $intd_{\delta(0),0} \times [-1, 1]$  and  $intd_{\theta(0),0} \times [-1, 1]$  are disjoint from  $e_\gamma^l \times I$ . Hence each component of  $e_\lambda^l \times I \cap e_\gamma^l \times I$  is either in  $e_\lambda^l \times I - e_\lambda^l \times I \cap d_{j_0}^l \times [-1, 1]$  or in  $d_{j_0}^l \times [-1, 1]$ . By the argument in Case 4, the lemma holds.

Case 6.  $Max\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\} \geq \lambda > \gamma$ .

By Definition 5.3.2,  $e_\lambda^l \times I \cap e_\gamma^l \times I = S_1 \cup S_2$  where  $S_1 = (e_\lambda^l \times I - e_\lambda^l \times I \cap (h_1 \cup h_2) \times I) \cap (e_\gamma^l \times I - e_\gamma^l \times I \cap (h_1 \cup h_2) \times I)$  and  $S_2 = H(e_\lambda^l \times I \cap (h_1 \cup h_2) \times I) \cap H(e_\gamma^l \times I \cap (h_1 \cup h_2) \times I)$ . Since  $H$  is a homeomorphism, the lemma holds. Q.E.D.

**Lemma 5.5.5.**  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^l$  is a abstract tree, and  $F^{l+1}$  is generated by  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e_\gamma^l$ .

**Proof.** By Lemma 5.4.4 and Lemmas 5.5.1-5.5.4, we only need to prove that  $\cup_f E_f^{l+1} \cup_{\gamma} e_\gamma^l$  is a abstract tree. By Lemma 5.4.4,  $\partial_1 e_{m^{l+1}}^l \subset \partial E_{f_{\delta(0),0}}^{l+1}$  and  $\partial_2 e_{m^{l+1}}^l \subset \partial E_{f_{\theta(0),0}}^{l+1}$ . By the argument in Lemma 4.4.5, the lemma holds. Q.E.D.

## 5.6 Properties of $d_j'$

**Lemma 5.6.1.** Suppose that  $j \in \{1, \dots, n\} - m(l+1)$ ,  $\gamma \in m(l+1)$  and  $j \notin I(w_\gamma, l+1)$ . If  $j < \gamma$ , then  $d_j'$  is disjoint from  $e_\gamma^l \times I$ .

**Proof.** Suppose that  $j < \gamma$  and  $j \notin I(w_\gamma, l+1)$ . By Proposition 4(3),  $d_j^l$  is disjoint from  $e_\gamma^l \times I$  for  $\gamma \in m(l)$ . Without loss of generality, we may assume that  $s(w_{\gamma_0}) = -$ . Now there are three cases:

Case 1.  $\gamma \neq m^{l+1}$ ,  $j > Max\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

Now  $\gamma > j > Max\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . By Definition 5.3.2 and Lemma 5.4.4,  $e_\gamma^l \times I = e_\lambda^l \times I$  and  $d_j' = d_j^l$ . Hence  $d_j'$  is disjoint from  $e_\gamma^l \times I$ .

Case 2.  $\gamma = m^{l+1}$ .

Since  $j < m^{l+1}$ ,  $j < \gamma_0$ . By Lemma 5.1.2(5),  $d_j' = d_j^l$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . By Lemma 5.1.4(2),  $d_j'$  is disjoint from  $\partial V_{l+1}^l$ . Hence  $d_j'$  is disjoint from  $e_{m^{l+1}}^l \times I \subset d_{j_0}^l \times [-2, 2] \cup \partial V_{l+1}^l \times I$ .

Case 3.  $j < \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

Now by Lemma 5.1.2(5) and Lemma 5.1.4(2),  $d_j^l$  is disjoint from  $\partial V_{l+1}^l \times I$  and  $d_{j_0}^l \times [-2, 2]$ . By Definition 5.3.2,  $e'_\gamma \times I \subset e_\gamma^l \times I \cup d_{j_0}^l \times [-2, 2] \cup \partial V_{l+1}^l \times I$ . Hence  $d'_j = d_j^l$  is disjoint from  $e'_\gamma \times I$ . Q.E.D.

**Lemma 5.6.2.** Suppose that  $j \in \{1, \dots, n\} - m(l+1)$ ,  $\gamma \in m(l+1)$ . If  $j > \gamma$ , then each component of  $d'_j \cap e'_\gamma \times I$  is a core  $c \subset \text{int}d_j^l$  of  $e'_\gamma \times (0, 1)$ .

**Proof.** Suppose that  $j > \gamma$ . By Proposition 4(2), each component of  $d'_j \cap e_\gamma^l \times I$  is a core  $c \subset \text{int}d_j^l$  of  $e_\gamma^l \times (0, 1)$  for  $\gamma \in m(l)$ . Without loss of generality, we may assume that  $s(w_{\gamma_0}) = -$ . Now there are four cases:

Case 1.  $\gamma \neq m^{l+1} > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

By Definition 5.3.2 and Lemma 5.4.4,  $d'_j = d_j^l$  and  $e'_\gamma \times I = e_\gamma^l \times I$ . Hence the lemma holds.

Case 2.  $\gamma = m^{l+1}$ .

By Definition 5.3.2,  $e'_{m^{l+1}} \times I = B_3$ . By Lemma 5.1.4,  $d_j^l \cap \partial V_{l+1}^l \times I = d_j^l \cap c_{l+1}^l \times I$ . By Lemma 5.1.2 and Lemma 3.1.6, each component of  $d_j^l \cap \partial V_{l+1}^l \times I$  is an arc  $c \subset \text{int}d_j^l$  which lies either in  $d_{j_0}^l \times [-1, 1]$  or in  $c_{l+1}^l \times I - d_{j_0}^l \times [-2, 2]$ . If  $c$  lies in  $d_{j_0}^l \times [-1, 1]$ , then  $c$  is disjoint from  $e'_{m^{l+1}} \times I$ . If  $c$  lies in  $c_{l+1}^l \times I - d_{j_0}^l \times [-2, 2]$ , then  $c$  intersects  $e'_{m^{l+1}} \times I$  in a core of  $e'_{m^{l+1}} \times (0, 1)$ . Hence the lemma holds.

Case 3.  $\gamma < j < \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ .

Now  $j < m^{l+1}$ . By Lemma 5.1.4(2),  $d'_j = d_j^l$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Lemma 5.1.2(5),  $d'_j$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . By Definition 5.3.2,  $e'_\gamma \times I \subset e_\gamma^l \times I \cup d_{j_0}^l \times [-2, 2] \cup \partial V_{l+1}^l \times I$ . Hence  $d'_j \cap e'_\gamma \times I = d_j^l \cap e_\gamma^l \times I$ , and the lemma holds.

Case 4.  $j > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\} \geq \gamma$ .

Now by Lemma 5.1.2(5),  $d'_j = d_j^l$  is disjoint from  $d_{j_0}^l \times [-2, -1] \cup [1, 2]$ . By Lemma 5.1.2(6),  $\text{int}d_{\delta(0),0} \times [-2, 2]$  and  $\text{int}d_{\theta(0),0} \times [-2, 2]$  are disjoint from  $e_\gamma^l \times I$ . Hence each component of  $d'_j \cap e_\gamma^l \times I$  is either in  $d_{j_0}^l \times [-1, 1]$  or disjoint from  $d_{j_0}^l \times [-2, 2]$ . By Lemma 3.1.6, each component of  $d'_j \cap c_{l+1}^l \times I$  is  $c \subset \text{int}d_j^l$  with  $\partial_1 c \subset c_{l+1}^l \times \{1\}$  and  $\partial_2 c \subset c_{l+1}^l \times \{-1\}$ . By Lemma 5.1.4,  $e_\gamma^l \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . Now  $d'_j \cap e'_\gamma \times I = S_1 \cup S_2$ , where

$S_1 = (d_j^l - d_{j_0}^l \times [-1, 1]) \cap e_\gamma^l \times I$  and  $S_2 = d_j^l \cap \partial V_{l+1}^l \times I \cap H(e_\gamma^l \times I \cap (h_1 \cup h_2) \times [-2, 2])$ .

Hence the lemma holds. Q.E.D.

**Lemma 5.6.3.** Suppose that  $j \notin L(c_{l+1}^l)$ . Then

- (1)  $d_j^l$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e_\gamma^l \times I$ .
- (2) For each  $i \geq l+2$ ,  $d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I$  intersects  $c_i^{l+1}$  in at most one point. Furthermore,  $d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I$  intersects  $c_i^{l+1}$  in one point if and only if  $d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I$  intersects  $c_i^l$  in one point.

**Proof.** Without loss of generality, we may assume that  $s(w_{\gamma_0}) = -$ .

By Lemma 3.1.2, in  $F^l$ ,  $d_j^l = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l \cup_{i=1}^{\theta(j)-1} e_{\gamma_{i,j}}$ , where  $d_{j,f_{i,j}}^l$  is a properly embedded arc in  $E_{f_{i,j}}^l$  which is disjoint from  $\cup_{\gamma < j} \text{inte}_\gamma^l \times I$  and  $e_{\gamma_{i,j}}$  is a core of  $e_{\gamma_{i,j}}^l \times (0, 1)$  for some  $j > \gamma_{i,j} \in m(l)$ . Furthermore,  $f_{i,j} \neq f_{r,j}$  and  $\gamma_{i,j} \neq \gamma_{r,j}$  for  $i \neq r$ .

By Lemma 5.1.2,  $d_j^l$  is disjoint from  $d_{j_0}^l \times [-2, -1] \cup [1, 2]$ . Hence  $e_{\gamma_{i,j}}$  is also a core of  $e_{\gamma_{i,j}}^l \times (0, 1)$ . Since  $j \notin L(c_{l+1}^l)$ ,  $d_{j,f_{i,j}}^l$  is disjoint from  $c_{l+1}^l \times I$  and  $\partial V_{l+1}^l \times I$  even if  $f_{i,j} = 0$ . Furthermore, if  $f_{i,j} = 0$ , then  $d_{j,0}^l$  lies in in one of  $E_0$  and  $E_{l+1}$ . See Definition 5.2.1. By Definition 5.3.2,  $e_\gamma^l \times I \subset e_\gamma^l \times I \cup d_{j_0}^l \times [-2, -1] \cup [1, 2] \cup \partial V_{l+1}^l \times I$ . Hence if  $\gamma < j$ , then  $d_{j,f_{i,j}}^l$  is disjoint from  $\text{inte}_\gamma^l \times I$ . Now each component of  $d_j^l \cap (\cup_{\gamma < j} e_\gamma^l \times I)$  is  $e_{\gamma_{i,j}}$  for  $1 \leq i \leq \theta(j) - 1$ . By Lemmas 5.6.1 and 5.6.2, (1) holds.

Now  $d_j^l - \cup_{\gamma < j} \text{inte}_\gamma^l \times I = \cup_{i=1}^{\theta(j)} d_{j,f_{i,j}}^l$ . Since  $c_i^{l+1} = (c_i^l - d_{j_0}^l \times [-2, 2]) \cup H(c_i^l \cap d_{j_0}^l \times [-2, 2])$ . Since  $j \neq j_0$ ,  $d_{j,f_{i,j}}^l$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . Hence (2) holds. Q.E.D.

**Lemma 5.6.4.** Suppose that  $j = j_\alpha \in L(c_{l+1}^l)$  and  $\alpha \neq 0$ . Then

- (1)  $d_{j_\alpha}^l - \text{inta}_\alpha = \cup_{\delta(\alpha)}^{\delta(0,\alpha)-1} d_{i,\alpha} \cup_{\delta(\alpha)}^{\delta(0,\alpha)-1} e_{i,\alpha} \cup_{\theta(0,\alpha)+1}^{\delta(\alpha)} d_{i,\alpha} \cup_{\theta(0,\alpha)+1}^{\theta(\alpha)} e_{i,\alpha}$ .
- (2)  $d_{i,\alpha}$  is disjoint from  $\cup_{\gamma < j_\alpha} \text{inte}_\gamma^l \times I$ .
- (3)  $d_{i,\alpha}$  intersects  $c_i^{l+1}$  in one point if and only if  $d_{i,\alpha}$  intersects  $c_i^l$  in one point for  $\delta(\alpha) \leq i \leq \delta(0, \alpha) - 1$  or  $\theta(0, \alpha) + 1 \leq i \leq \theta(\alpha)$ .
- (4) For  $\gamma < j_\alpha$ , each component of  $d_{j_\alpha}^l \cap e_\gamma^l \times I$  is contained either in  $d_{j_\alpha}^l - \text{inta}_\alpha$  or in  $\text{inta}_\alpha$ .

**Proof.** By Definition 3.5.1,  $a_\alpha = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$ . Hence  $d_{0,\alpha} \subset a_\alpha$ ,  $d_{i,\alpha}$  is disjoint from  $c_{l+1}^l \times I$  and  $\partial V_{l+1}^l \times I$  for  $\delta(0) \leq i \leq \delta(0, \alpha) - 1$  or  $\theta(0, \alpha) + 1 \leq i \leq \theta(\alpha)$ .

Now by the argument in Lemma 5.6.3, (1), (2) and (3) hold.

Suppose that  $\delta(0, \alpha) \leq -1$ . Then  $f_{\delta(0, \alpha), \alpha} \neq 0$ . Hence  $d_{\delta(0, \alpha), \alpha} \subset E_{f_{\delta(0, \alpha), \alpha}}^{l+1}$  is disjoint from  $c_{l+1}^l \times I$  and  $\partial V_{l+1}^l \times I$ . By the proof of Lemma 5.6.3,  $int d_{\delta(0, \alpha), \alpha}$  is disjoint from  $e'_\gamma \times I$  for  $\gamma < j_\alpha$ .

Now we assume that  $\delta(0, \alpha) = \theta(0, \alpha) = 0$ . Since  $\alpha \neq 0$ , By Proposition 4(3) and Lemma 3.3.1,  $d_{i, \alpha}$  is disjoint from  $d_{j_0}^l \times [-2.5, 2.5]$ . Now  $a_\alpha = a^1 \cup a^2 \cup a^3$  where  $a^1$  is an arc in  $E_0 \subset E_0^{l+1}$ ,  $a^2$  is an arc in  $c_{l+1}^l \times I - d_{j_0}^l \times [-2.5, 2.5]$  (by Lemma 5.1.2(7)),  $a^3$  is an arc in  $E_{l+1} \subset E_{l+1}^{l+1}$ . By Definition 5.2.3 and Definitions 5.3.2 and 5.3.3,  $e'_\gamma \times I \subset e_\gamma^l \times I \cup \partial V_{l+1}^l \times I \cup d_{j_0}^l \times [-2, 2]$ . Hence  $inta^1, inta^3$  are disjoint from  $e'_\gamma \times I$  for  $\gamma < j_\alpha$ . Thus (4) holds. Q.E.D.

## 5.7 Properly embedded disks in $\mathcal{V}_-$ and $\mathcal{W}_-$

In this section, we shall prove the following Lemma:

**Lemma 5.7.1.** There are two sets of pairwise disjoint disks  $\{V_i^{l+1} \mid i \geq l+2 \text{ with } s(v_i) = -\}$  properly embedded in  $\mathcal{V}_-$  and  $\{W_j' \mid j \in \{1, \dots, n\} - m(l+1) \text{ with } s(w_j) = -\}$  properly embedded in  $\mathcal{W}_-$  such that

- (1)  $\partial V_i^{l+1} \cap F^{l+1} = c_i^{l+1} \cup_{r \in I(v_i, l+1)} c_i^{l+1}, \partial W_j' \cap F^{l+1} = d_j' \cup_{r \in I(w_j, l+1)} d_j';$
- (2)  $V_i^{l+1} \cap W_j' = V_i^{l+1} \cap W_j' \cap F^{l+1}.$

**Proof.** Suppose that  $i \geq l+2$  and  $s(v_i) = -$ . Then, by Proposition 6,  $V_i^l$  is a properly embedded disk in  $\mathcal{V}_-$  such that  $V_i^l \cap F^l = c_i^l \cup_{i \in I(v_i, l)} c_r^l$ . By Lemma 5.1.4,  $V_i^l$  is disjoint from  $\partial V_{l+1}^l \times I$ . By assumption,  $s(v_{l+1}) = -$ . By Lemma 2.2.4,  $l+1 \notin I(v_i, l)$ . By Definition 2.3.1,  $I(v_i, l+1) = I(v_i, l)$ . Now let  $C_i = (\partial V_i^l - c_i^l \cup_{i \in I(v_i, l)} c_r^l) \cup c_i^{l+1} \cup_{i \in I(v_i, l+1)} c_r^{l+1}$ . By Lemma 5.4.3(4),  $c_i^{l+1}$  and  $c_r^{l+1}$  are obtained by doing band sums with copies  $\partial V_{l+1}^l$  to  $c_i^l$  and  $c_r^l$ . Hence  $C_i$  bounds a disk in  $\mathcal{V}_-$ , denoted by  $V_i^{l+1}$ . Since  $F^{l+1} \subset F^l \cup \partial V_{l+1}^l \times I$ . Hence  $V_i^{l+1} \cap F^{l+1} = c_i^{l+1} \cup_{i \in I(v_i, l+1)} c_r^{l+1}.$

Suppose now that  $j \in \{1, \dots, n\} - m(l+1)$  and  $s(w_j) = -$ . Then, by Proposition 6,  $W_j^l$  is a properly embedded disk in  $F^l$  such that  $W_j^l \cap F^l = d_j^l \cup_{r \in I(w_j, l)} d_r^l$ . By Lemma 5.1.4,  $W_j^l - d_j^l \cup_{r \in I(w_j, l)} d_r^l$  is disjoint from  $\partial V_{l+1}^l \times I$ . We denote by  $W_j'$  the disk  $W_j^l$ . There are

two cases:

Case 1.  $\gamma_0 \neq \emptyset$ . See Definition 3.2.1.

Since  $j \notin m(l+1)$  and  $\gamma_0 \in m(l)$ , by Lemma 2.2.5,  $j_0 \notin I(w_j, l)$ . By Definition 2.3.1,  $I(w_j, l+1) = I(w_j, l)$ . Since  $d'_j = d_j^l$  and  $d'_r = d_r^l$ ,  $W'_j \cap F^{l+1} = d'_j \cup_{r \in I(w_j, l+1)} d'_r$ . By Proposition 6,  $\partial V_{l+1}^l - c_i^l \cup_{i \in I(v_i, l)} c_r^l$  is disjoint from  $W_j^l - d_j^l \cup_{r \in I(w_j, l)} d_r^l$ . Hence  $V_i^{l+1} \cap W'_j = (c_i^{l+1} \cup_{i \in I(v_i, l+1)} c_r^{l+1}) \cap (d'_j \cup_{r \in I(w_j, l+1)} d'_r)$ . Hence (2) holds.

Case 2.  $\gamma_0 = \emptyset$ .

In this case,  $I(w_j, l+1) = I(w_j, l) - \{m^{l+1}\}$ . Now by Proposition 6,  $d_{j_0}^l$  is disjoint from  $\partial V_i^l - c_i^l \cup_{i \in I(v_i, l)} c_r^l$ . By Definition 5.2.1,  $d_{j_0}^l$  is disjoint from  $F^{l+1}$ . By Definition 5.2.3,  $d_{j_0}^l$  is disjoint from  $c_i^{l+1}$ . Hence  $W'_j \cap F^{l+1} = d'_j \cup_{r \in I(w_j, l+1)} d'_r$  and  $V_i^{l+1} \cap W'_j = (c_i^{l+1} \cup_{i \in I(v_i, l+1)} c_r^{l+1}) \cap (d'_j \cup_{r \in I(w_j, l+1)} d'_r)$ . Q.E.D.

## 5.8 The proofs of Propositions 4-6

In this section, we shall first construct  $d_j^{l+1}, e_\gamma^{l+1} \times I$  from  $d'_j$  and  $e'_\gamma \times I$  for  $j \in \{1, \dots, n\} - m(l+1)$  and  $\gamma \in m(l+1)$ . Then we shall prove Propositions 4-6 for the case:  $k = l+1$  and  $s(v_{l+1}) = -$ .

### **Construction(\*\*).**

Since  $j_\alpha > j_0 = m^{l+1}$  for  $\alpha \neq 0$ ,  $\gamma_\alpha > m^{l+1}$  if  $\gamma_\alpha \neq \emptyset$ . By Lemma 5.1.2(7) and Lemma 5.1.3(6),  $d_{j_0}^l \times [-2.5, 2.5]$  is disjoint from  $(a_\alpha \times I)_{\gamma_\alpha}$ . By Definitions 3.2.1 and Lemma 5.4.4,  $a_\alpha \subset d'_j = d_j^l$ ,  $(a_\alpha \times I)_{\gamma_\alpha} \subset (d'_{j_\alpha} \times I)_{\gamma_\alpha} \subset e'_{\gamma_\alpha} \times I = e_{\gamma_\alpha}^l \times I$  if  $s(w_{\gamma_\alpha}) = -$ , and  $(a_\alpha \times I)_{\gamma_\alpha} = a_\alpha \subset d'_{j_\alpha}$  if  $s(w_{\gamma_\alpha}) = +$  or  $\gamma_\alpha = \emptyset$ .

Without loss of generality, we may assume that, in  $F^l$ ,  $d_{j_0}^l \times [-2.5, 2.5] \cap D_{0,\alpha}^* \subset d_{j_0}^l \times [0, 2.5]$  for  $\alpha > 0$ , and  $d_{j_0}^l \times [-2.5, 2.5] \cap D_{0,\alpha}^* \subset d_{j_0}^l \times [-2.5, 0]$  for  $\alpha < 0$ . See Lemma 3.5.5.

By Definition 5.2.2(2) and Definition 5.3.1(2), in  $F^{l+1}$ ,  $d_{j_0}^l \times [-2.5, -1.5]$  is a disk such that  $\partial d_{j_0}^l \times [-2.5, -2] = \partial d_{j_0}^l \times [-2, -1.5]$ , and  $d_{j_0}^l \times [1.5, 2.5]$  is a disk such that  $\partial d_{j_0}^l \times [1.5, 2] = \partial d_{j_0}^l \times [2, 2.5]$ .

For  $\alpha > 0$ , let  $C_\alpha$  be a simple closed curve in  $d_{j_0}^l \times [1.5, 2.5]$  satisfying the following conditions:



(1)  $C_\alpha$  intersects  $d_{j_0}^l \times [2, 2.5]$  in a core  $d_{j_0}^l \times \{t_\alpha\}$  of  $d_{j_0}^l \times [2, 2.5]$  where  $t_\alpha \in (2, 2.5)$ ,  $C_\alpha$  intersects  $e'_{m^{l+1}} \times I$  in a core of  $e'_{m^{l+1}} \times (0, 1)$  lying in  $d_{j_0}^l \times (1.6, 1.7)$ , say  $e_{m^{l+1}, \alpha}$ .

(2) If  $0 < \alpha < \beta$ , then  $t_\alpha > t_\beta$ .

(3) If  $\alpha \neq \beta$ , then  $C_\alpha \cap C_\beta = \emptyset$ . See Figure 25.

For  $\alpha < 0$ , let  $C_\alpha$  be a simple closed curve in  $d_{j_0}^l \times [-2.5, -2]$  satisfying the following conditions:

(4)  $C_\alpha$  intersects  $d_{j_0}^l \times [-2.5, -2]$  in a core  $d_{j_0}^l \times \{t_\alpha\}$  of  $d_{j_0}^l \times [-2.5, -2]$  where  $t_\alpha \in (-2.5, -2)$ ,  $C_\alpha$  intersects  $e'_{m^{l+1}} \times I$  in a core of  $e'_{m^{l+1}} \times (0, 1)$  lying in  $d_{j_0}^l \times (-1.7, -1.6)$ , say  $e_{m^{l+1}, \alpha}$ .

(5) If  $\beta < \alpha < 0$ , then  $t_\beta > t_\alpha$ .

(6) If  $\alpha \neq \beta$ , then  $C_\alpha \cap C_\beta = \emptyset$ . See Figure 25.

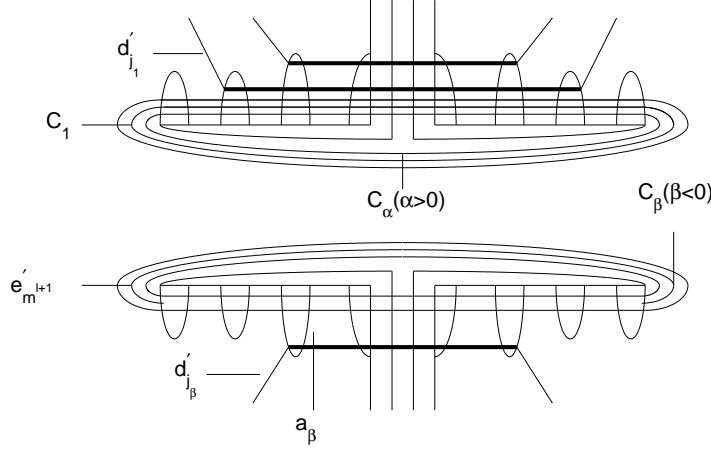


Figure 25

Now  $C_\alpha$  bounds a disk in  $d_{j_0}^l \times [-2.5, -1.5] \subset F^{l+1}$  for  $\alpha < 0$ , and  $C_\alpha$  bounds a disk in  $d_{j_0}^l \times [1.5, 2.5] \subset F^{l+1}$  for  $\alpha > 0$ .

Note that  $a_\alpha = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,\alpha} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,\alpha}$ , and  $a_\alpha^0 \times \{t_\alpha\} = \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} d_{i,0} \times \{t_\alpha\} \cup_{i=\delta(0,\alpha)}^{\theta(0,\alpha)} e_{i,0} \times \{t_\alpha\}$ . Since  $a_\alpha$  and  $a_\alpha^0 \times \{t_\alpha\}$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . By Definition 5.2.3, Definitions 5.3.2 and 5.3.3,  $\partial_1 a_\alpha, \partial_1 a_\alpha^0 \times \{t_\alpha\} \subset E_{f_{\delta(0,\alpha),i}}^{l+1}$  and  $\partial_2 a_\alpha, \partial_2 a_\alpha^0 \times \{t_\alpha\} \subset E_{f_{\theta(0,\alpha),i}}^{l+1}$ . In particular, if  $\delta(0, \alpha) = \theta(0, \alpha) = 0$ . By Lemma 3.5.8, Definition 5.2.3 and Definition 5.3.3, we may assume that  $E_{f_{\delta(0,\alpha),i}}^{l+1} = E_0^{l+1}$  and  $E_{f_{\theta(0,\alpha),i}}^{l+1} = E_{l+1}^{l+1}$ .

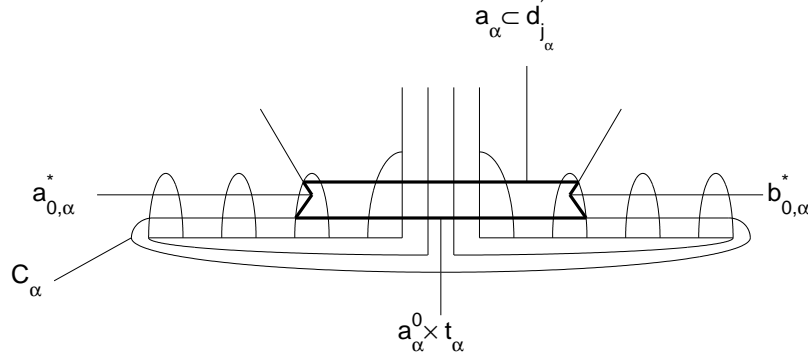


Figure 26

Now let  $a_{0,\alpha}^*$  be an arc in  $E_{f_{\delta(0,\alpha),i}}^{l+1}$  connecting  $\partial_1 a_\alpha$  to,  $\partial_1 a_\alpha^0 \times \{t_\alpha\}$  such that

(7)  $a_{0,\alpha}^* - d_{j_0}^l \times [-2.5, 2.5] = a_{0,\alpha} - d_{j_0}^l \times [-2.5, 2.5]$  and  $a_{0,\alpha}^* \cap d_{j_0}^l \times [-2.5, 2.5]$  is an arc lying in  $d_{\delta(0,\alpha),0} \times [-2.5, -2] \cup [2, 2.5]$ .

Now let  $b_{0,\alpha}^*$  be an arc in  $E_{f_{\theta(0,\alpha),i}}^{l+1}$  connecting  $\partial_2 a_\alpha$  to,  $\partial_2 a_\alpha^0 \times \{t_\alpha\}$  such that

(8)  $b_{0,\alpha}^* - d_{j_0}^l \times [-2.5, 2.5] = b_{0,\alpha} - d_{j_0}^l \times [-2.5, 2.5]$  and  $b_{0,\alpha}^* \cap d_{j_0}^l \times [-2.5, 2.5]$  is an arc lying in  $d_{\theta(0,\alpha),0} \times [-2.5, -2] \cup [2, 2.5]$ .

Where  $a_{0,\alpha}, b_{0,\alpha}$  are as in Lemma 3.5.5. See Figure 26.

By Lemma 5.1.4(1) and the proof of Lemma 4.3.5,  $a_{0,\alpha}^* \cup b_{0,\alpha}^*$  is disjoint from  $c_{l+1}^l \times I$  and  $\partial V_{l+1}^l \times I$ . Furthermore,  $(a_{0,\alpha}^* \cup b_{0,\alpha}^*) \cap (a_{0,\beta}^* \cup b_{0,\beta}^*) = \emptyset$  for  $\alpha, \beta \neq 0, \alpha \neq \beta$ .

Now let  $b_\alpha = a_{0,\alpha}^* \cup (C_\alpha - a_\alpha^0 \times \{t_\alpha\}) \cup b_{0,\alpha}^*$ , and  $b_\alpha \times I$  be a neighborhood of  $b_\alpha$  in  $F^{l+1}$  satisfying the following conditions:

(9)  $(\partial b_\alpha) \times I = ((\partial a_\alpha) \times I)_{\gamma_\alpha}$  if  $s(w_{\gamma_\alpha}) = -$ .

(10)  $\partial b_\alpha = ((\partial a_\alpha) \times I)_{\gamma_\alpha}$  if  $s(w_{\gamma_\alpha}) = +$  or  $\gamma_\alpha = \emptyset$ .

(11)  $b_\alpha \times I \subset d_{j_0}^l \times (-2.5, -2) \cup (2, 2.5) \cup (-1.7, -1.6) \cup (1.6, 1.7)$ .

(12) For  $\alpha \neq \beta$ ,  $b_\alpha \times I \cap b_\beta \times I = \emptyset$ . Q.E.D.(Construction(\*\*))

**Lemma 5.8.1.** (1) If  $d_j' \cap (a_{0,\beta}^* \times I \cup b_{0,\beta}^* \times I) \neq \emptyset$ , then  $d_j' \cap (a_{0,\beta}^* \times I \cup b_{0,\beta}^* \times I) \subset \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}$ . Furthermore, either  $j = j_\lambda$ , or  $j > \gamma_\lambda$  for some  $\lambda \neq 0$ .

(2)  $e_\gamma' \times I \cap (a_{0,\beta}^* \times I \cup b_{0,\beta}^* \times I) \subset \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}$ . If  $e_\gamma' \times I \cap (inta_{0,\beta}^* \times I \cup intb_{0,\beta}^* \times I) \neq \emptyset$ , then  $e_\gamma' \times I \cap (inta_{0,\beta}^* \times I \cup intb_{0,\beta}^* \times I) \subset \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}$ . Furthermore,  $\gamma \geq \gamma_\lambda$  for some  $\lambda \neq 0$ .

**Proof.** By the proof of Lemma 4.3.5 and Lemma 5.1.4(1),  $a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I$  is disjoint from  $\partial V_{l+1}^l \times I$ . By Construction(\*\*)(7) and (8),  $inta_{0,\alpha}^* \times I \cup intb_{0,\alpha}^* \times I$  is disjoint from  $d_{j_0}^l \times [-2, 2]$ . Hence  $inta_{0,\alpha}^* \times I \cup intb_{0,\alpha}^* \times I$  is disjoint from  $e'_{m^{l+1}} \times I \subset d_{j_0}^l \times [-2, 2] \cup \partial V_{l+1}^l \times I$ . Suppose  $j \notin m(l+1)$  and  $\gamma \in m(l)$ . By Definitions 5.2.3, 5.3.2 and 5.3.3,  $d_j' \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I) = d_j' \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I)$  and  $e_\gamma' \times I \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I) = e_\gamma^l \times I \cap (a_{0,\alpha}^* \times I \cup b_{0,\alpha}^* \times I)$ . By Lemma 4.3.5, the lemma holds. Q.E.D.

**Lemma 5.8.2.**  $b_\alpha = a_{0,\alpha}^* \cup_{i=\delta(0)}^{\delta(0,\alpha)-1} d_{i,0} \times \{t_\alpha\} \cup_{i=\delta(0)}^{\delta(0,\alpha)-1} e_{i,0} \times \{t_\alpha\} \cup e_{m^{l+1},\alpha} \cup_{i=\theta(0,\alpha)+1}^{\theta(0)} d_{i,0} \times \{t_\alpha\} \cup_{i=\theta(0,\alpha)+1}^{\theta(0)} e_{i,0} \times \{t_\alpha\} \cup b_{0,\alpha}^*$  satisfying the following conditions:

- (1)  $intd_{i,0} \times \{t_\alpha\}$  is disjoint from  $e_\gamma' \times I$  for  $\gamma \in m(l+1)$ .
- (2)  $e_{i,0} \times \{t_\alpha\}$  is a core of  $e_\gamma' \times (0, 1)$ .
- (3)  $e_{m^{l+1},\alpha}$  is a core of  $e_{m^{l+1}}' \times (0, 1)$ .

Where  $e_{m^{l+1},\alpha}$  is as in Construction(\*\*)(1).

**Proof.** (1) By Definition 3.5.1,  $d_{0,0} \subset a_\alpha^0$  for each  $\alpha \neq 0$ . Hence  $f_{i,0} \neq 0$  for  $\delta(0) \leq i \leq \delta(0,\alpha) - 1$  or  $\theta(0,\alpha) + 1 \leq i \leq \theta(0)$ . Hence  $d_{i,0} \times [-2.5, 2.5]$  is disjoint from  $c_{l+1}^l \subset E_0^l$ . Since  $t_\alpha \in [-2.5, -2] \cup [2, 2.5]$ , by Definitions 5.2.3, 5.3.2 and Lemmas 5.1.2(8), 5.1.3(4),  $intd_{i,0} \times \{t_\alpha\}$  is disjoint from  $e_\gamma' \times I$  for each  $\gamma \in m(l+1)$ .

(2) By Definitions 5.2.3, 5.3.2,  $e_{i,0} \times \{t_\alpha\}$  is a core of  $e_\gamma' \times (0, 1)$ .

(3) follows from Construction(\*\*), Definitions 5.2.3 and 5.3.2. Q.E.D.

**Lemma 5.8.3.** (1) If  $\gamma \neq m^{l+1} > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , then  $C_\alpha - inta_\alpha^0 \times \{t_\alpha\}$  is disjoint from  $e_\gamma' \times I$ .

(2)  $(C_\alpha - inta_\alpha^0 \times \{t_\alpha\}) \cap e_{m^{l+1}}' \times I = e_{m^{l+1},\alpha}$ .

(3) If  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ , then each component of  $(C_\alpha - inta_\alpha^0 \times \{t_\alpha\}) \cap e_\gamma' \times I$  is a core of  $e_\gamma' \times (0, 1)$ .

**Proof.** (1) Suppose that  $\gamma \neq m^{l+1} > \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . Since  $(C_\alpha - inta_\alpha^0 \times \{t_\alpha\}) \subset d_{j_0}^l \times (-2.5, -1.5) \cup (1.5, 2.5)$ , by Lemma 5.1.2(2) and (4), Lemma 5.1.3(2),  $e_\gamma^l \times I$  is disjoint from  $C_\alpha - inta_\alpha^0 \times \{t_\alpha\}$ . By Lemma 5.4.4,  $e_\gamma' \times I = e_\gamma^l \times I$ . Hence (1) holds.

(2) By Construction(\*\*)(1),  $C_\alpha - inta_\alpha^0 \times \{t_\alpha\} = \cup_{i=\delta(0)}^{\delta(0,\alpha)-1} d_{i,0} \times \{t_\alpha\} \cup_{i=\delta(0)}^{\delta(0,\alpha)-1} e_{i,0} \times \{t_\alpha\} \cup e_{m^{l+1},\alpha} \cup_{i=\theta(0,\alpha)+1}^{\theta(0)} d_{i,0} \times \{t_\alpha\} \cup_{i=\theta(0,\alpha)+1}^{\theta(0)} e_{i,0} \times \{t_\alpha\}$ . Since  $d_{0,0} \subset a_\alpha^0$ , by Lemma

5.1.4(4),  $C_\alpha - \text{inta}_\alpha^0 \times \{t_\alpha\} - e_{m^{l+1}, \alpha}$  is disjoint from  $c_{l+1}^l \times I$  and  $\partial V_{l+1}^l \times I$ . Furthermore,  $C_\alpha - \text{inta}_\alpha^0 \times \{t_\alpha\} - e_{m^{l+1}, \alpha} \subset d_{j_0}^l \times (-2.5, -2) \cup (2, 2.5)$ , and  $e'_{m^{l+1}} \times I \subset d_{j_0}^l \times [-2, 2] \cup \partial V_{l+1}^l \times I$ . Hence (2) holds.

(3) Suppose that  $\gamma \leq \text{Max}\{\gamma_{i,0} \mid \delta(0) \leq i \leq \theta(0)\}$ . Then  $\gamma < j_0 = m^{l+1}$ . By (2),  $C_\alpha - \text{inta}_\alpha^0 \times \{t_\alpha\} - e_{m^{l+1}, \alpha}$  is disjoint from  $c_{l+1}^l \times I$ . By Lemma 5.5.2,  $\text{inte}_{m^{l+1}, \alpha}$  is disjoint from  $e'_\gamma \times I$ . By Definition 5.2.3 and Definition 5.3.3,  $(C_\alpha - \text{inta}_\alpha^0 \times \{t_\alpha\} - \text{inte}_{m^{l+1}, \alpha}) \cap e'_\gamma \times I = (C_\alpha - \text{inta}_\alpha^0 \times \{t_\alpha\} - \text{inte}_{m^{l+1}, \alpha}) \cap e_\gamma^l \times I$ . By Lemma 5.1.2 and Lemma 5.1.3, (3) holds. Q.E.D.

**Definition 5.8.4.** (1) If  $s(w_{\gamma_\alpha}) = -$ , then let  $H_\alpha$  be a homeomorphism from  $(a_\alpha \times I)_{\gamma_\alpha}$  to  $b_\alpha \times I$  such that  $H_\alpha$  is an identifying map on  $((\partial a_\alpha) \times I)_{\gamma_\alpha}$ .

(2) If  $s(w_\gamma) = +$  or  $\gamma_\alpha = \emptyset$ , let  $H_\alpha$  be a homeomorphism from  $a_\alpha = (a_\alpha \times I)_{\gamma_\alpha}$  to  $b_\alpha$  such that  $H_\alpha$  is an identifying map on  $\partial a_\alpha$ .

**Definition 5.8.5.** (1) For  $j \notin m(l+1)$ , let  $d_j^{l+1} = (d'_j - \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\alpha \neq 0} H_\alpha(d'_j \cap (a_\alpha \times I)_{\gamma_\alpha})$ .

(2) For  $\gamma \in m(l+1)$ , if  $\gamma \leq m^{l+1}$ , let  $e_\gamma^{l+1} \times I = e'_\gamma \times I$ ; if  $\gamma > m^{l+1}$ , let  $e_\gamma^{l+1} \times I = (e'_\gamma \times I - \cup_{\alpha \neq 0} (a_\alpha \times I)_{\gamma_\alpha}) \cup_{\alpha \neq 0} H_\alpha(e'_\gamma \times I \cap (a_\alpha \times I)_{\gamma_\alpha})$ .

**Lemma 5.8.6.**  $d_j^{l+1}$  is isotopic to  $d'_j$  in  $F^{l+1} \subset \partial_+ \mathcal{V}_-$ .

**Proof.** By Lemma 3.5.5,  $a_{0,\alpha}^* \cup a_\alpha \cup a_\alpha^0 \times \{t_\alpha\} \cup b_{0,\alpha}^*$  bounds a disk in  $F^{l+1}$ . See Figure 26. By Construction(\*\*),  $C_\alpha$  bounds a disk in  $F^{l+1}$ . Hence  $a_\alpha$  is isotopic to  $b_\alpha$ . By Lemma 5.6.1 and Lemma 5.6.2, each component of  $d'_j \cap (a_\alpha \times I)_{\gamma_\alpha}$  is a core of  $(a_\alpha \times I)_{\gamma_\alpha}$  even if  $j = j_\alpha$ , say  $c_j$ . By Definitions 5.8.4 and 5.8.5,  $H_\alpha(c_j)$  is a core of  $b_\alpha \times I$ . Hence  $d_j^{l+1}$  is isotopic to  $d'_j$ . Q.E.D.

Now we prove Propositions 4-6 for the case:  $k = l+1$  and  $s(v_{l+1}) = -$ .

**The proofs of Propositions 4-6.** By Lemma 5.4.4, Lemmas 5.5.1-5.5.5,  $F^{l+1}$  is generated by the abstract tree  $\cup_f E_f^{l+1} \cup_{\gamma \in m(l+1)} e'_\gamma$  satisfying the following conditions:

- (1) If  $s(w_\lambda) = +$ , then  $\text{inte}'_\lambda \times I$  is disjoint from  $\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I$ .
- (2) If  $s(w_\lambda) = -$ , then  $\text{inte}'_\lambda \times I \cup (\cup_f E_f^{l+1} \cup_{\gamma < \lambda} e'_\gamma \times I) = \cup_{r \in I(w_\lambda, l+1)} (d'_r \times I)_\lambda$ .

By Lemma 5.6.1, if  $j < \gamma$  and  $j \notin I(w_\gamma, l+1)$ , then  $d'_j$  is disjoint from  $e'_\gamma \times I$ .

By Lemma 5.6.2, each component of  $d'_j \cap e'_\gamma \times I$  is a core of  $e'_\gamma \times (0, 1)$  for  $j \notin m(l+1)$ ,  $\gamma \in m(l+1)$  and  $j > \gamma$ .

By Lemma 5.6.3,  $d'_j$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j} e'_\gamma \times I$  for  $j \notin L(c_{l+1}^l)$ .

By Lemma 5.4.3,  $c_i^{l+1}$  is an arc properly embedded in  $F^{l+1}$  which lies in one of  $E_f^{l+1}$  for some  $f$ ;  $\partial c_i^{l+1} \cap e'_\gamma \times I = \emptyset$  for each  $\gamma \in m(l+1)$ , and  $\partial c_i^{l+1} \cap d'_j = \emptyset$  for each  $j \in \{1, 2, \dots, n\} - m(l+1)$ .

Now if we take place of  $c_i^{l+1}, d_j^{l+1}, E_f^{l+1}, e_\gamma^{l+1} \times I, F^{l+1}$  with  $c_i^{l+1}, d'_j, E_f^{l+1}, e'_\gamma \times I, F^{l+1}$ , then Proposition 4 holds except that

- (i)  $d'_{j_\alpha}$  is regular in  $\cup_f E_f^{l+1} \cup_{\gamma < j_\alpha} e'_\gamma \times I$ ;
- (ii)  $d'_{j_\alpha} - \cup_{\gamma < j_\alpha} \text{inte}'_\gamma \times I$  intersects  $c_i^{l+1}$  in at most one point.

Where  $j_\alpha \in L(c_{l+1}^l)$ .

Now by Lemmas 5.6.4, 5.8.1, 5.8.2, 5.8.3 and the argument in Chapter 4, Proposition 4 holds for the case:  $k = l+1$  and  $s(v_{l+1}) = -$ .

By Lemmas 5.6.3, 5.6.4, 5.8.2 and the proof of Proposition 5 in Section 4.6, Proposition 5 holds for the case:  $k = l+1$  and  $s(v_{l+1}) = -$ .

Proposition 6 follows from Lemmas 5.8.6, 5.7.1 and the proof of Proposition 6 in Section 4.6. Q.E.D.

## 6 The proofs of Propositions 1-3

Now we prove Propositions 1-3 for  $k = l+1$  under the assumptions that Propositions 1-6 hold for  $k \leq l$ . Then we finish the proofs of Propositions 1-6.

**The Proofs of Propositions 1-3.** By Lemma 3.1.5,  $L(v_i^l) = L(c_i^l)$  and  $L(d_j^l) = L(w_j^l)$ . Hence  $m^{l+1} = \text{Min}L(c_{l+1}^l) = \text{Min}L(v_{l+1}^l)$ . Now there are two cases:

Case 1.  $s(v_{l+1}) = -$ .

Now by Lemma 5.1.1(4),  $s(w_{m^{l+1}}) = +$ . Now by Remark 3.5.9 and the argument in Chapter 4, Propositions 1-3 hold.

Case 2.  $s(v_{l+1}) = +$ .

By Remark 3.5.9 and the argument in Chapter 5, Propositions 1-3 hold. Q.E.D.

Now Propositions 1-6 hold for  $0 \leq k \leq m$ . Hence Theorem 1 is true.

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